Chapter 3: Deterministic Continuous-Time Optimal Control

- Continuous time optimal control
- Calculus of variation
- Hamilton-Jacobi-Bellman (HJB)
- Minimum principle
- Applications

Continuous time optimal control

Problem formulation

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)), 0 \leq t \leq T \\ x(0) &= x_0 \end{aligned}$$

Find control $u \in U$ so

$$\min \quad h(x(T)) + \int_0^T g(x(t), u(t))dt$$

This gives state trajectory $x^u(t)$

Ex: (i). $h(x(T)) = |x(T) - \bar{x}|$ (ii). $g(x, u) = 1$

Calculus of variation

The Branchistocrone problem (Bernoulli 1696)
Idea how to solve the optimization problem

$$\begin{aligned} I(\alpha) &= \int_{t_1}^{t_2} L(t, q(t), \dot{q}(t))dt \\ \text{Let } q(t,\alpha) &= q(t) + \alpha\eta(t), \qquad \eta(t_1) = \eta(t_2) = 0 \\ \text{Optimum} &\rightarrow \alpha = 0 \\ \frac{dI}{d\alpha}(\alpha) &= \int (\frac{\partial L}{\partial q}\frac{\partial q}{\partial \alpha} + \frac{\partial L}{\partial \dot{q}}\frac{\partial \dot{q}}{\partial \alpha})dt = \{\text{p.i.}\} = \\ &= \int \frac{\partial L}{\partial q}\eta(t) + \left[\frac{\partial L}{\partial \dot{q}}\eta(t)\right] - \int \eta(t)\frac{d}{dt}\frac{\partial L}{\partial \dot{q}}dt = \\ &= \int (\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}})\eta(t)dt \end{aligned}$$

Calculus of variation

Optimum for $\alpha = 0$:

$$\frac{dI}{d\alpha}(0) = 0 \,\&\, \eta - \text{arbitrary}$$

Euler-Lagrange equation $\boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0}$

Remark: $q \Leftrightarrow q_j,\ j = 1, 2, \ldots, N$

Dynamic Programming (DP)

Deterministic DP-algorithm (recall chapter 1)

$$
\begin{aligned}
J_N(x_N) &= g_N(x_N) \\
J_k(x_k) &= \min_{u_k} g_k(x_k,u_k) + J_{k+1}(x_{k+1},u_{k+1}),\ k=0,1,\dots,N-1
\end{aligned}
$$

Theorem $J^*(x_0) = J_0(x_0)\ \forall x_0$ using the DP-algorithm

We want to use the DP-algorithm on the optimization problem

Hamilton-Jacobi-Bellman (HJB)

Continuous cost function: $h(x(T)) + \int_0^T g(x(t),u(t))dt$

Discretize the time interval $[0,T] \Rightarrow \delta = \frac{T}{N}$

$$
\begin{aligned}
x_k &= x(k\delta), u_k = u(k\delta), k=0,1,\dots,N \\
\dot{x}(t) &= f(x(t),u(t)) \Rightarrow x_{k+1} = x_k + f(x_k,u_k)\delta
\end{aligned}
$$

Discretized cost function

$$h(x_N) + \sum_{k=0}^{N-1} g(x_k,u_k)\delta$$

HJB–Informal derivation

Dynamic Programming (DP) applied to the discretized function

$$\tilde{J}^*(N\delta,x) = h(x)$$

$$\tilde{J}^*(k\delta,x) = \min_{u\in U} g(x,u)\delta + \tilde{J}^*((k+1)\delta, x+f(x,u)\delta), k=0,1,\dots,N-1$$

Taylor expansion (assume $\tilde{J}^*$-differentiable)

$$
\begin{aligned}
&\tilde{J}^*((k+1)\delta, x+f(x,u)\delta) \approx \\
&\approx \tilde{J}^*(k\delta,x) + \nabla_t\tilde{J}^*(k\delta,x)\delta + \nabla_x\tilde{J}^*(k\delta,x)^T f(x,u)\delta
\end{aligned}
$$

Substitute into DP-equation and let $\delta \to 0$ yields the HJB-eq

$$0 = \min_u \left[g(x,u) + \nabla_t\tilde{J}^*(t,x) + \nabla_x\tilde{J}^*(t,x)^T f(x,u)\right], \forall t,x$$

HJB–partial diff. equation

Sufficiency theorem
Suppose V(t,x)–continuous differentiable solution to HJB

$$
\begin{aligned}
\min_u \left[g(x,u) + \nabla_t V(t,x) + \nabla_x V(t,x)^T f(x,u)\right] &= 0, \qquad \forall t,x \\
V(T,x) &= h(x), \qquad \forall x
\end{aligned}
$$

Then V is unique and $V(t,x) = J^*(t,x), \forall t,x$ (optimal cost-to-go)

LQ–Application

Linear system with quadratic cost

$$\begin{aligned} \dot{x} = Ax(t) + Bu(t) &= f(x,u) \\ h(x(T)) + \int_0^T g(x,u)dt &= x(T)^T Q_T x(T) + \int_0^T x(t)^T Qx(t) + u(t)^T Ru(t)dt \end{aligned}$$

HJB :

$$\min_u \left[g(x,u) + \nabla_t V(t,x) + \nabla_x V(t,x)^T f(x,u) \right] = 0, \qquad \forall t, x$$
$$V(T,x) = h(x)$$

Try a solution: $V = x^T K(t) x(t)$ which yields
Optimal control : $u^* = -R^{-1} B^T K(t) x$ and the Riccati-equation:

$$\dot{K}(t) = -K(t)A - A^T K(t) + K(t) B R^{-1} B^T K(t) - Q$$
$$K(T) = Q_T$$

HJB–How to proceed

Difficult to solve HJB-equation.

Lemma: If $\mu^*(t,x) = \underset{u}{argmin} F(t,x,u) \, \forall t, x$ then

$$\begin{aligned} \nabla_t \{\min_u F\} &= \nabla_t F(t,x,\mu^*) \\ \nabla_x \{\min_u F\} &= \nabla_x F(t,x,\mu^*) \end{aligned}$$

Using the lemma on HJB-eq $\Rightarrow$ differential eq.
Introducing $p(t) \Rightarrow u^*(t) = \underset{u \in U}{\text{argmin}} \, g(x,u) + p^T f(x,u)$

The method is summarized in the so called Minimum Principle

The Minimum Principle

Define the Hamiltonian H

$$H(x,u,p) = g(x,u) + p^T f(x,u)$$

where the adjoint equation

$$\dot{p}(t) = -\nabla_x H(x^*(t), u^*(t), p(t))$$

Boundary condition

$$p(T) = \nabla h(x^*(T))$$

And

$$u^*(t) = \underset{u \in U}{\text{argmin}} \, H(x^*(t), u, p(t))$$
$$H(x^*(t), u*(t), p(t)) = constant$$

Minimum principle–remarks

Necessary for optimality i.e. all opt. trajectories satisfies the principle.
However, if satisfied it is not necessarily optimal.

Negative sign (instable), numerical problems

$$\dot{p}(t) = -\nabla_x H(x^*(t), u^*(t), p(t))$$

H-constant since

$$\dot{H} = \frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial p}\dot{p} + \frac{\partial H}{\partial u}\dot{u} = 0$$

Application–Bang-Bang

Special case: Linear, minimum time, controllable ,constrainted control

$$\dot{x} = Ax + Bu,\ u_{min} \le u(t) \le u_{max},\ \mathcal{R}^{nxn}$$

Theorem Control signal switches between u_{min}, u_{max}.
Corollary If A:s eigenvalues $\in \mathcal{R} \Rightarrow n-1$ switches (max)

Extensions

Consider the following variations of the minimum principle

- Fixed terminal state
- Free initial state
- Free terminal time
- Time-varying system and cost
- Singular problems

Extensions

Fixed terminal state
$x(0) = x_0, x(T) = X_T$ known $\Rightarrow$ p(T)-unknown
Free initial state
$x(0)$ is subject to optimization $\Rightarrow p(0) = 0$
Free terminal time
T is subject to optimization $\Rightarrow H = 0$ (constant)
Time-varying system and cost
Convert to time-invariant problem
Note: H–not constant along optimal trajectory
Singular problems
$x^*(t), p(t)$ such that $H(x^*(t), u, p(t), t)$ independent of u (interval)

Example: Minimum time

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t)$$

Want to minimize T, i.e. $\int_0^T 1 dt$, when $-1 \le u \le 1$
Hamiltonian function : $H = g + p^T f = 1 + Ax + Bu$
Boundary conditions: $x(0) = x_0$-given and $x(T) = [0\,0]^T$
Minimum principle:

$$u^*(t) = \underset{-1 \le u \le 1}{argmin}(1 + p_1(t)x_2^*(t) + p_2(t)u) = \begin{cases} 1 & \text{, if } p_2 < 0 \\ -1 & \text{, if } p_2 \ge 0 \end{cases}$$

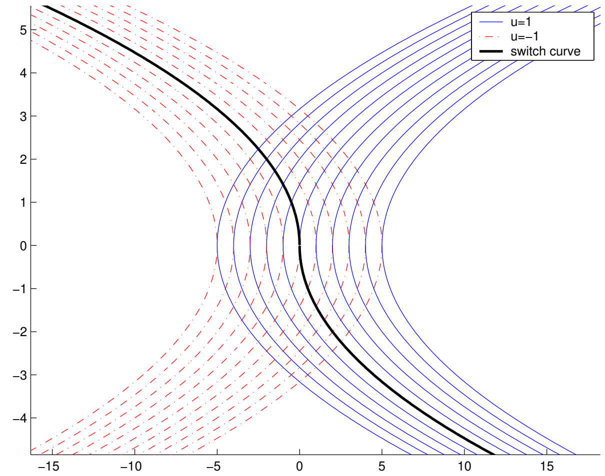
Example: Minimum time –continued

The adjoint eq: $\dot{p_1}(t) = 0$ and $\dot{p_2}(t) = -p_1(t)$
So

$$\begin{aligned} p_1(t) &= c_1 \\ p_2(t) &= c_2 - c_1 t \end{aligned}$$

Note: $p_2(t) = 0 \forall t$ not possible (since $H = 0$)

Optimal control & switch-curve: $\frac{dx_2}{dx_1} = \frac{\dot{x_2}}{\dot{x_1}} \Rightarrow x_1 = \frac{1}{2} u x_2^2 + C$



Suggested Reading

Calculus of variation and optimal control

- Classical mechanics (Goldstein)
- Calculus of variation and optimal control (Leitmann)
- Reglerteori