Filtering and Identification

Day 3 - Lecture 1: Interpretation and use of Kalman Filtering

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Overview

- **The Square Root KF** — checking optimality?
- Case Study: Level estimation in liquid tanks
- The Kalman filter for LTI systems
- Application 1: Estimating an unknown constant
- Application 2: Estimating an unknown input

The square root solution to SLS

The given WLS problem:

$$\min_x \nu^T \nu \quad \text{subject to} \quad \begin{bmatrix} \bar{x} \\ y \end{bmatrix} = \begin{bmatrix} I_n \\ F \end{bmatrix} x + \begin{bmatrix} -P^{1/2} & 0 \\ 0 & L \end{bmatrix} \nu$$

with $\nu \sim (0, I_n)$, is equivalent to:

$$\min_x (\nu')^T (\nu') \quad \text{subject to} \quad T_\ell \begin{bmatrix} \bar{x} \\ y \end{bmatrix} = T_\ell \begin{bmatrix} I_n \\ F \end{bmatrix} x + T_\ell \begin{bmatrix} -P^{1/2} & 0 \\ 0 & L \end{bmatrix} T_r \underbrace{(T_r^T \nu)}_{\nu'}$$

with T_ℓ and T_r resp. invertible and orthogonal. Furthermore, $\nu' \sim (0, I_n)$

The application of T_ℓ and T_r

A (non-singular) transformation T_ℓ that does the job is:

$$T_\ell = \begin{bmatrix} F & -I \\ I & 0 \end{bmatrix} \quad \text{applied as} \quad T_\ell \begin{bmatrix} \bar{x} \\ y \end{bmatrix} = T_\ell \begin{bmatrix} I_n \\ F \end{bmatrix} x + T_\ell \begin{bmatrix} -P^{1/2} & 0 \\ 0 & L \end{bmatrix} T_r T_r^T \nu$$

transforms the contraint set into:

$$\begin{bmatrix} F\bar{x} - y \\ \bar{x} \end{bmatrix} = \begin{bmatrix} \boxed{0} \\ I \end{bmatrix} x + \begin{bmatrix} -FP^{1/2} & -L \\ -P^{1/2} & 0 \end{bmatrix} T_r T_r^T \nu$$

Important is the introduced “zero”! T_r follows from an LQ fac:

$$\begin{bmatrix} F\bar{x} - y \\ \bar{x} \end{bmatrix} = \begin{bmatrix} \boxed{0} \\ I \end{bmatrix} x + \begin{bmatrix} R & 0 \\ G & P_{new}^{1/2} \end{bmatrix} \begin{bmatrix} \kappa \\ \delta \end{bmatrix}$$

Recall the KF signal generating model

State reconstruction is considered for the following signal generating model (SGM):

$$\begin{aligned} x(k+1) &= A(k)x(k) + B(k)u(k) + w(k) \\ y(k) &= C(k)x(k) + v(k) \quad \text{with} \quad x(0) \sim \Big(\widehat{x}(0|-1), P(0|-1)\Big) \quad \text{and} \end{aligned}$$

$$\begin{bmatrix} v(k) \\ w(k) \end{bmatrix} \sim \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} R(k) & S(k)^T \\ S(k) & Q(k) \end{bmatrix} \geq 0 \right) \ \& \ \text{white} \qquad R(k) > 0$$

The problem was to find a minimum variance reconstruction of the state $x(k), x(k+1)$, etc.

Recall the WLS problem to derive the KF

$$\min_{x(k),x(k+1)} \mu(k)^T \mu(k) \quad \text{subject to}$$

$$\begin{bmatrix} \widehat{x}(k|k-1) \\ y(k) \\ -B(k)u(k) \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ C(k) & 0 \\ A(k) & -I_n \end{bmatrix} \begin{bmatrix} x(k) \\ x(k+1) \end{bmatrix}$$

$$+ \begin{bmatrix} P(k|k-1)^{1/2} & 0 & 0 \\ 0 & R(k)^{1/2} & 0 \\ 0 & X(k) & Q_x(k)^{1/2} \end{bmatrix} \underbrace{\begin{bmatrix} \widetilde{x}(k) \\ \widetilde{v}(k) \\ \widetilde{w}(k) \end{bmatrix}}_{\mu(k)}$$

Recall the WLS problem to derive the KF

$$\min_{x(k),x(k+1)} \mu(k)^T \mu(k) \quad \text{subject to}$$

$$T_\ell \begin{bmatrix} \widehat{x}(k|k-1) \\ y(k) \\ -B(k)u(k) \end{bmatrix} = T_\ell \begin{bmatrix} I_n & 0 \\ C(k) & 0 \\ A(k) & -I_n \end{bmatrix} \begin{bmatrix} x(k) \\ x(k+1) \end{bmatrix}$$

$$+T_\ell \begin{bmatrix} P(k|k-1)^{1/2} & 0 & 0 \\ 0 & R(k)^{1/2} & 0 \\ 0 & X(k) & Q_x(k)^{1/2} \end{bmatrix} T_r T_r^T \underbrace{\begin{bmatrix} \widetilde{x}(k) \\ \widetilde{v}(k) \\ \widetilde{w}(k) \end{bmatrix}}_{\mu(k)}$$

Using an invertible transformation T_ℓ and an orthogonal transformation T_r, similar to the square root solution to the WLS problem, we can transform the constraint into (see next slide) $\cdots$

The Square Root Covariance Filter (SRCF)

$$\begin{bmatrix} C(k)\widehat{x}(k|k-1) - y(k) \\ \widehat{x}(k|k-1) \\ B(k)u(k) + A(k)\widehat{x}(k|k-1) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ I_n & 0 \\ 0 & I_n \end{bmatrix} \begin{bmatrix} x(k) \\ x(k+1) \end{bmatrix} + \begin{bmatrix} R^e(k)^{1/2} & 0 & 0 \\ \star & \star & \star \\ G(k) & P(k+1|k)^{1/2} & 0 \end{bmatrix} \underbrace{\begin{bmatrix} \nu(k) \\ \widetilde{x}(k+1) \\ \widetilde{w}'(k) \end{bmatrix}}_{\mu'(k)}$$

where it still holds that $E[\mu'(k)\Big(\mu'(k)\Big)^T] = I$!

The Square Root Covariance Filter (SRCF)

$$\begin{bmatrix} C(k)\widehat{x}(k|k-1) - y(k) \\ \widehat{x}(k|k-1) \\ B(k)u(k) + A(k)\widehat{x}(k|k-1) - G(k)\nu(k) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ I_n & 0 \\ 0 & I_n \end{bmatrix} \begin{bmatrix} x(k) \\ x(k+1) \end{bmatrix} + \begin{bmatrix} R^e(k)^{1/2} & 0 & 0 \\ \star & \star & \star \\ 0 & P(k+1|k)^{1/2} & 0 \end{bmatrix} \underbrace{\begin{bmatrix} \nu(k) \\ \widetilde{x}(k+1) \\ \widetilde{w}'(k) \end{bmatrix}}_{\mu'(k)}$$

where it still holds that $E[\mu'(k)\Big(\mu'(k)\Big)^T] = I$! From which the generalized covariance representation of $x(k+1)$ follows:

$$\begin{aligned} x(k+1) &= B(k)u(k) + A(k)\hat{x}(k|k-1) + G(k)R^e(k)^{-1/2}\Big(y(k) - C(k)\hat{x}(k|k-1)\Big) \\ &\quad + P(k+1|k)^{1/2}\tilde{x}(k) \qquad \tilde{x}(k) \sim (0, I) \end{aligned}$$

The SRCF - conventional KF link

It can be shown (see book p. 145 - 150) that the SRCF update of the one-step ahead prediction of the state $x(k+1)$ given as:

$$\begin{aligned} x(k+1) &= B(k)u(k) + A(k)\hat{x}(k|k-1) + G(k)R^e(k)^{-1/2}\Big(y(k) - C(k)\hat{x}(k|k-1)\Big) \\ &\quad +P(k+1|k)^{1/2}\tilde{x}(k) \qquad \tilde{x}(k) \sim (0, I) \end{aligned}$$

is equal to the conventional KF update, given as:

$$\hat{x}(k+1|k) = B(k)u(k) + A(k)\hat{x}(k|k-1) + K(k)\Big(y(k) - C(k)\hat{x}(k|k-1)\Big)$$

with

$$\begin{aligned} K(k) &= (S(k) + A(k)P(k|k-1)C(k)^T)\Big(R(k) + C(k)P(k|k-1)C(k)^T\Big)^{-1} = G(k)R^e(k)^{-1/2} \\ P(k+1|k) &= A(k)P(k|k-1)A(k)^T + Q(k) \\ &\quad -(S(k) + A(k)P(k|k-1)C(k)^T)\Big(R(k) + C(k)P(k|k-1)C(k)^T\Big)^{-1}(S(k) + A(k)P(k| \\ &= P(k+1|k)^{1/2}P(k+1|k)^{T/2} \end{aligned}$$

Comparison SRCF - CKF

Computational Complexity: full calculations

SRCF	$\frac{7}{6}n^3 + n^2(\frac{5\ell}{2} + m) + n(\ell^3 + \frac{m^2}{2})$
CKF	$\frac{3}{2}n^3 + n^2(3\ell + \frac{m}{2}) + n(\frac{3\ell^2}{2} + m^2) + \frac{\ell^2}{6}$

Numerical Reliability

Exercise for this afternoon

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Checking optimality of the KF

The Kalman filter is optimal since it:

1. minimizes the state error covariance matrix (see application COS principle).
2. But how can we check that from the observations and quantities that are computable from the filter recursions.

Two consecutive SRCF updates

The prior and data at time instances k and $k+1$ are collected in the constraint:

$$\left[\begin{array}{c} \begin{bmatrix} \widehat{x}(k|k-1) \\ y(k) \\ -B(k)u(k) \end{bmatrix} \\ \hline y(k+1) \\ -B(k+1)u(k+1) \end{array}\right] = \left[\begin{array}{cc|c} \begin{bmatrix} I_n & 0 \\ C(k) & 0 \\ A(k) & -I_n \end{bmatrix} & & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \\ \hline \begin{matrix} 0 & C(k+1) \\ 0 & A(k+1) \end{matrix} & & \begin{matrix} 0 \\ -I_n \end{matrix} \end{array}\right] \left[\begin{array}{c} x(k) \\ x(k+1) \\ \hline x(k+2) \end{array}\right]$$

$$+ \left[\begin{array}{c|c} \begin{bmatrix} P(k|k-1)^{1/2} & 0 & 0 \\ 0 & R(k)^{1/2} & 0 \\ 0 & X(k) & Q_x(k)^{1/2} \end{bmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} \\ \hline \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{matrix} R(k+1)^{1/2} & 0 \\ X(k+1) & Q_x(k+1)^{1/2} \end{matrix} \end{array}\right] \left[\begin{array}{c} \widetilde{x}(k) \\ \widetilde{v}(k) \\ \widetilde{w}(k) \\ \hline \tilde{v}(k+1) \\ \tilde{w}(k+1) \end{array}\right]$$

The first SRCF update

$$
\left[\begin{array}{c}
\begin{bmatrix} C(k)\widehat{x}(k|k-1) - y(k) \\ \widehat{x}(k|k-1) \\ B(k)u(k) + A(k)\widehat{x}(k|k-1) - G(k)\nu(k) \end{bmatrix} \\ \hline
y(k+1) \\
-B(k+1)u(k+1)
\end{array}\right]
=
\left[\begin{array}{ccc}
\begin{bmatrix} 0 & 0 \\ I_n & 0 \\ 0 & I_n \end{bmatrix} & | & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \\ \hline
\begin{matrix} 0 & C(k+1) \\ 0 & A(k+1) \end{matrix} & | & \begin{matrix} 0 \\ -I_n \end{matrix}
\end{array}\right]
\left[\begin{array}{c} x(k) \\ x(k+1) \\ \hline x(k+2) \end{array}\right]
$$

$$
+ \left[\begin{array}{ccc}
\begin{bmatrix} R^e(k)^{1/2} & 0 & 0 \\ \star & \star & \star \\ 0 & 0 & P(k+1|k)^{1/2} \end{bmatrix} & | & \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} \\ \hline
\begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & | & \begin{matrix} R(k+1)^{1/2} & 0 \\ X(k+1) & Q_x(k+1)^{1/2} \end{matrix}
\end{array}\right]
\left[\begin{array}{c} \nu(k) \\ \tilde{w}'(k) \\ \widetilde{x}(k+1) \\ \hline \tilde{v}(k+1) \\ \tilde{w}(k+1) \end{array}\right]
$$

The second SRCF update (1)

The pre-array on which the next SRCF update is applied:

$$\left[\begin{array}{c} C(k)\widehat{x}(k|k-1)-y(k) \\ \widehat{x}(k|k-1) \\ \hline \left[\begin{array}{c} B(k)u(k)+A(k)\widehat{x}(k|k-1)-G(k)\nu(k) \\ y(k+1) \\ -B(k+1)u(k+1) \end{array}\right] \end{array}\right] = \left[\begin{array}{c|cc} 0 & 0 & 0 \\ I_n & 0 & 0 \\ \hline 0 & \multicolumn{2}{c}{\left[\begin{array}{cc} I_n & 0 \\ C(k+1) & 0 \\ A(k+1) & -I_n \end{array}\right]} \\ 0 & & \\ 0 & & \end{array}\right] \left[\begin{array}{c} x(k) \\ \hline x(k+1) \\ x(k+2) \end{array}\right]$$

$$+ \left[\begin{array}{cc|ccc} R^e(k)^{1/2} & 0 & 0 & 0 & 0 \\ \star & \star & \star & 0 & 0 \\ \hline 0 & 0 & \multicolumn{3}{c}{\left[\begin{array}{ccc} P(k+1|k)^{1/2} & 0 & 0 \\ 0 & R(k)^{1/2} & 0 \\ 0 & X(k+1) & Q_x(k+1)^{1/2} \end{array}\right]} \\ 0 & 0 & & & \\ 0 & 0 & & & \end{array}\right] \left[\begin{array}{c} \nu(k) \\ \tilde{w}'(k) \\ \hline \tilde{x}(k+1) \\ \tilde{v}(k+1) \\ \tilde{w}(k+1) \end{array}\right]$$

The second SRCF update (2)

The post-array:

$$\left[\begin{array}{c} C(k)\widehat{x}(k|k-1) - y(k) \\ \widehat{x}(k|k-1) \\ \hline \left[\begin{array}{c} C(k+1)\hat{x}(k+1|k) - y(k+1) \\ \hat{x}(k+1|k) \\ B(k+1)u(k+1) + A(k+1)\hat{x}(k+1|k) - \cdots \\ \cdots - G(k+1)\nu(k+1) \end{array}\right] \end{array}\right] = \left[\begin{array}{c|cc} 0 & 0 & 0 \\ I_n & 0 & 0 \\ \hline 0 & \multicolumn{2}{c}{\multirow{3}{*}{$\left[\begin{array}{cc} 0 & 0 \\ I_n & 0 \\ 0 & I_n \end{array}\right]$}} \\ 0 & & \\ 0 & & \end{array}\right] \left[\begin{array}{c} x(k) \\ \hline x(k+1) \\ x(k+2) \end{array}\right]$$

$$+ \left[\begin{array}{cc|ccc} R^e(k)^{1/2} & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & \star \\ \hline 0 & 0 & \multicolumn{3}{c}{\multirow{3}{*}{$\left[\begin{array}{ccc} R^e(k+1)^{1/2} & 0 & 0 \\ \star & \star & \star \\ 0 & 0 & P(k+2|k+1)^{1/2} \end{array}\right]$}} \\ 0 & 0 & & & \\ 0 & 0 & & & \end{array}\right] \left[\begin{array}{c} \nu(k) \\ \tilde{w}'(k) \\ \hline \nu(k+1) \\ \tilde{w}'(k+1) \\ \tilde{x}(k+2) \end{array}\right]$$

The optimality of the SRCF (KF)

The sequence of residuals $\nu(k), \nu(k+1), \nu(k+2), \cdots$ satisfy the following property by CONSTRUCTION!

$$E \begin{bmatrix} \nu(k) \\ \nu(k+1) \\ \nu(k+2) \\ \vdots \end{bmatrix} \begin{bmatrix} \nu(k)^T & \nu(k+1)^T & \nu(k+2)^T & \cdots \end{bmatrix} = \begin{bmatrix} I_\ell & 0 & 0 & \cdots \\ 0 & I_\ell & 0 & \\ 0 & 0 & I_\ell & \\ \vdots & & & \ddots \end{bmatrix}$$

The scaled sequence $\boxed{R^e(k)^{1/2}\nu(k) = y(k) - C(k)\hat{x}(k|k-1)}$ is called the KF Innovation sequence and this sequence is white when the model is perfect, the measurement noise covariance matrix $R(k) > 0$, etc.

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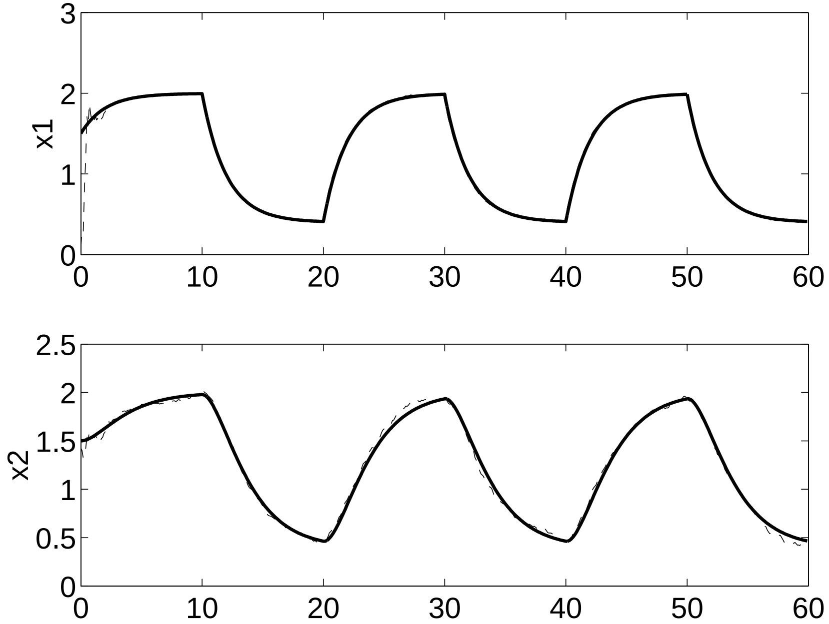
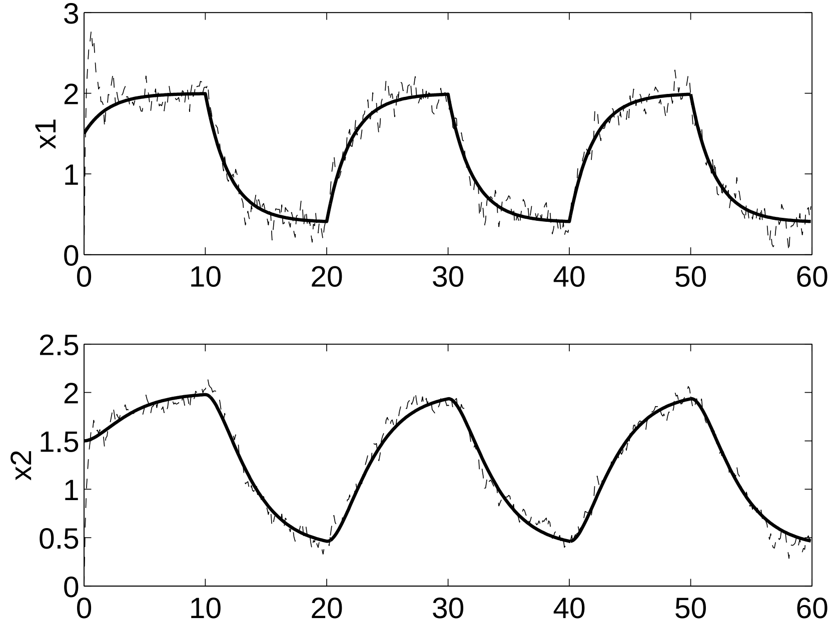
Example: Double tank with noisy data

Reliable Kalman Filter

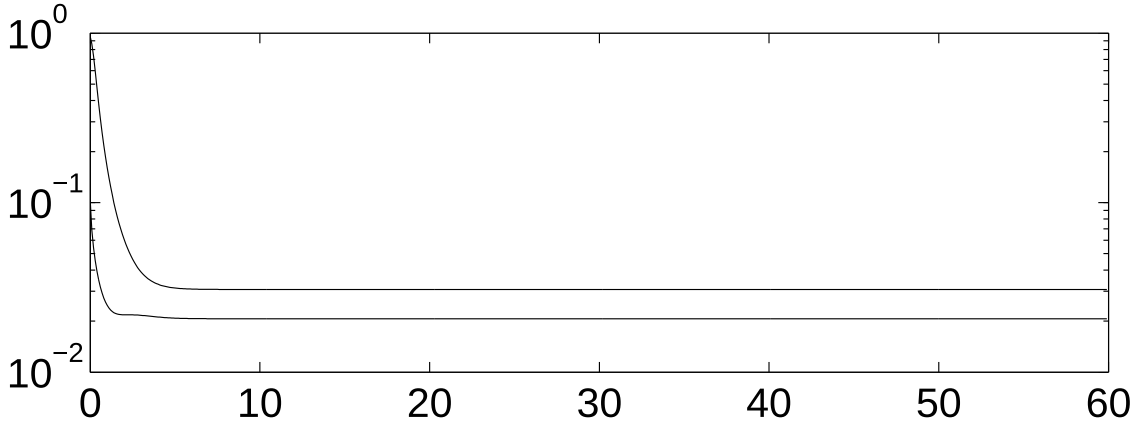


Asymptotic observer

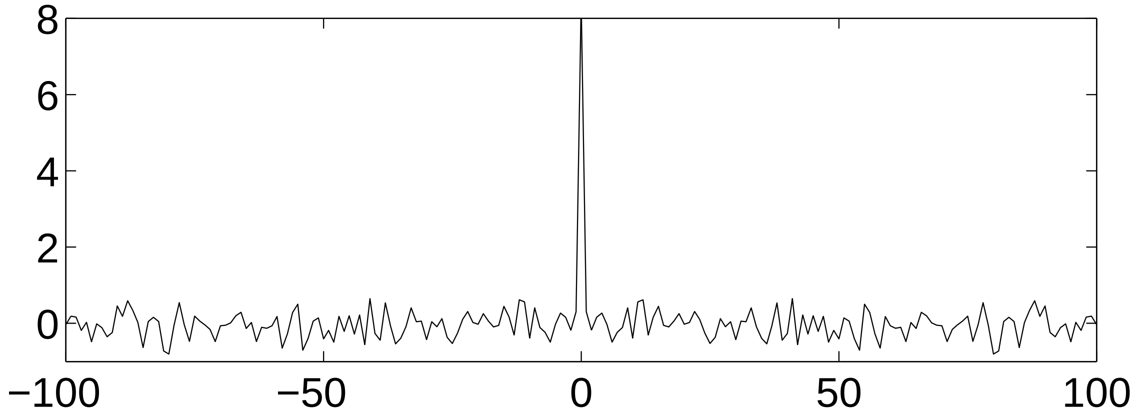


Example: Double tank with noisy data and SRCF

Time histories $P(k|k-1)(i,i)^{1/2}$



Auto-correlation $C_e(\tau)$



New Signal Generating Model

Let us denote the innovation sequence $y(k) - C(k)\hat{x}(k|k-1)$ by $e(k)$, then we can denote the Kalman filter recursion as:

$$\begin{aligned} \widehat{x}(k+1|k) &= A(k)\widehat{x}(k|k-1) + B(k)u(k) + K(k)e(k) \\ y(k) &= C(k)\widehat{x}(k|k-1) + e(k) \end{aligned}$$

with $e(k) \frown (0, R^e(k))$.

This is the so-called Innovation model

It is equivalent with the original SGM up to second order moment statitics, i.e. $E[y(k)y(k+\tau)^T]$, etc.

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The signal generating model and KF problem for the LTI case

State reconstruction is considered for the following signal generating model (SGM):

$$
\begin{aligned}
x(k+1) &= Ax(k) + Bu(k) + w(k) \\
y(k) &= Cx(k) + v(k) \quad \text{with } x(0) \frown \Big(\widehat{x}(0|-1), P(0|-1)\Big) \quad \text{and}
\end{aligned}
$$

$$
\begin{bmatrix} v(k) \\ w(k) \end{bmatrix} \frown \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} R & S^T \\ S & Q \end{bmatrix} \geq 0 \right) \quad R > 0
$$

Then the problem is to seek a linear estimate of $x(k)$ and $x(k+1)$ of the form,

$$
\begin{bmatrix} \hat{x}(k|k) \\ \hat{x}(k+1|k) \end{bmatrix} = M \begin{bmatrix} y(k) \\ -Bu(k) \end{bmatrix} + N\hat{x}(k|k-1)
$$

that are unbiased and minimum variance (UMV).

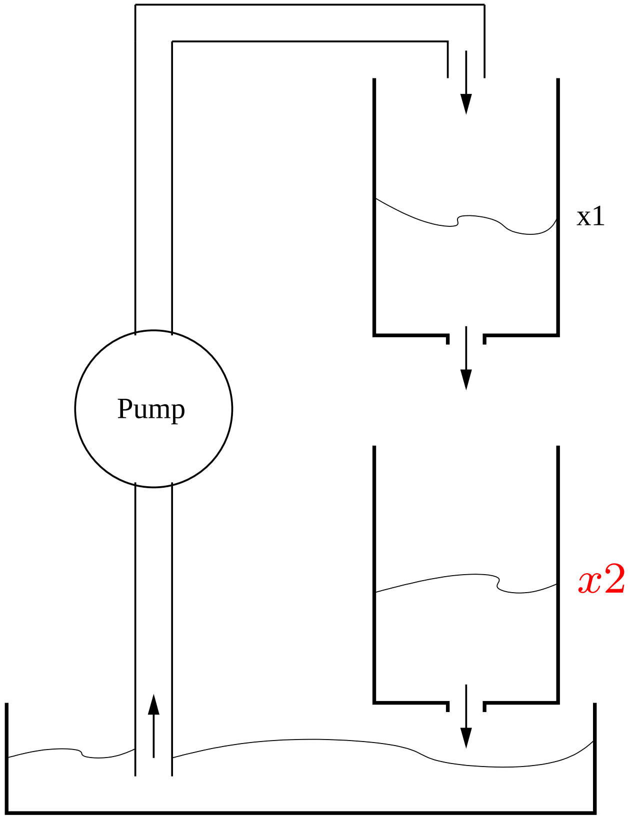
The stationary Kalman filter

Theorem: Consider the time-invariant signal generating model and statistical data. If the pair (A, C) is observable and the pair $(A, Q^{1/2})$ is controllable, then the covariance matrix of UMV estimate, i.e. $P(k|k-1)$, satisfies $\lim_{k\to\infty} P(k|k-1) = P > 0$, for any symmetric initial condition $P(0|-1) > 0$, where P satisfies

$$P = APA^T + Q - (S + APC^T)(CPC^T + R)^{-1}(S + APC^T)^T \quad \text{(DARE)}.$$

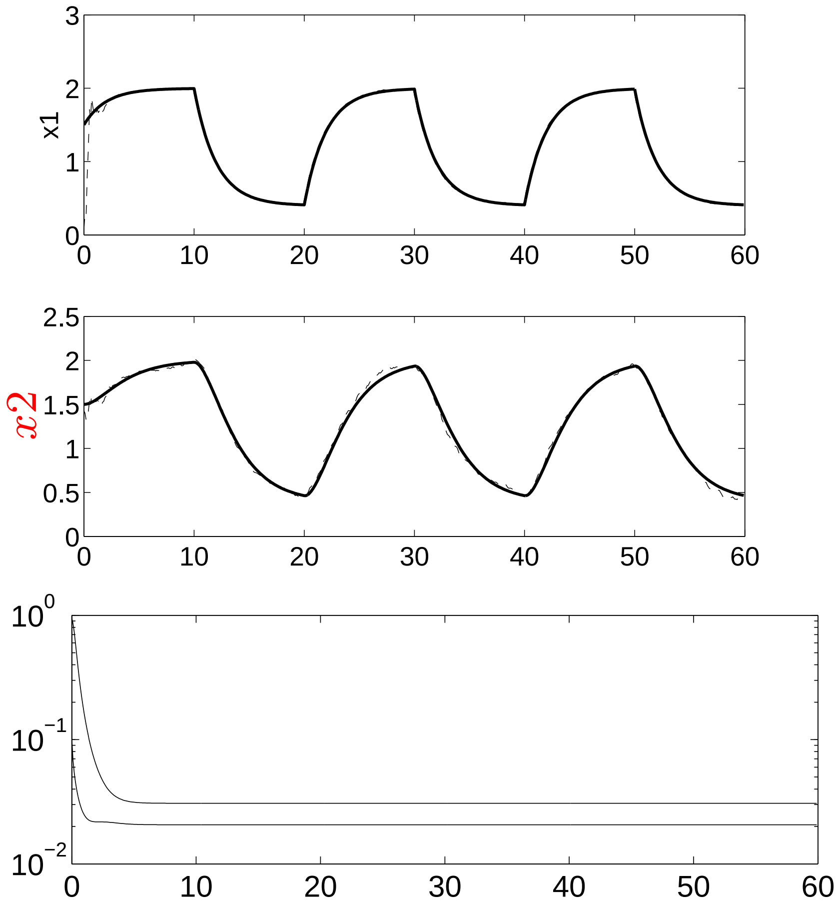
If this matrix P is used to define the Kalman gain matrix K as $K = (S + APC^T)(CPC^T + R)^{-1}$ then the matrix $A - KC$ is asymptotically stable.

Example: Double tank



Reconstruction and Time histories

$P(k|k-1)(i,i)$



Comp_Obs_SRCF.m

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Example: Estimating a constant

Consider improving a measurement of an 'almost' constant temperature $x(k)$.

$$\begin{aligned} x(k+1) &= x(k) + w(k) \\ y(k) &= x(k) + v(k) \end{aligned}$$

If we assume that $S \equiv 0$ and $Q \neq 0$, then the DARE reads:

$$P = P + Q - P(P+R)^{-1}P, \qquad P \in \mathbb{R}$$

or equivalently: $P^2 - QP - QR = 0$.

Interpretation process- and measurement statistics

1. The covariance matrix R of the measurement noise is a "measure" of the uncertainty of the sensor used.
2. The covariance matrix Q represents the variance of the "external" unknown input $w(k)$. The latter is used to represent either unknown inputs (physical) or the "confidence" in the state equation part (artificial).

Are there conditions by which filtering the output yields more accurate preditions of the output then obtained with the measurements?

Towards more accurate "virtual" sensing

The equation $P^2 - QP - QR = 0$ has two solutions:

$$
\begin{aligned}
P^+ &= \frac{Q + \sqrt{Q^2 + 4QR}}{2} \geq 0 \\
P^- &= \frac{Q - \sqrt{Q^2 + 4QR}}{2} \leq 0
\end{aligned}
$$

The value P^+ (the maximal root) is strictly smaller then R provided that $R > 2Q$.

The “roots” of the DARE

For P^+, the stationary Kalman filter has update equation:

$$\widehat{x}(k{+}1|k) = \left(1 - \frac{P^+}{P^+ + R}\right)\widehat{x}(k|k{-}1) + \frac{P^+}{P^+ + R}y(k)$$

which is a stable filter since

$$0 < 1 - \frac{P^+}{P^+ + R} < 1$$

If we take the negative root P^- (minimal), the stationary Kalman filter is anti-stable.

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Application 2: Estimating an unknown input

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