Preferably hand in your solutions as a single PDF file that also includes your m-files. Each student taking part in the exam has to submit one solution set.

Numerical Robustness Conventional Kalman Filter

Part 1: Matlab experiment

The conventional Kalman filter has been derived in Chapter 5 and is given for the LTI SGM:

$$
\begin{aligned}
x(k+1) &= Ax(k) + Bu(k) + w(k) \quad E[w(k)w(\ell)^T] = Q\Delta(k-\ell) \\
y(k) &= Cx(k) + v(k) \quad E[v(k)w(\ell)^T] = 0 \quad E[v(k)v(\ell)^T] = R\Delta(k-\ell)
\end{aligned}
$$

by the following recursions:

$$
\begin{aligned}
R^\epsilon(k) &= R + CP(k|k-1)C' \\
K(k) &= P(k|k-1)C'(R^\epsilon(k))^{-1} \\
P(k+1|k) &= AP(k|k-1)A' + Q - AP(k|k-1)C'(R^\epsilon(k))^{-1}CP(k|k-1)A' \\
\hat{x}(k+1|k) &= A\hat{x}(k|k-1) + Bu(k) + AK(y(k) - C\hat{x}(k|k-1))
\end{aligned}
$$

Especially in the early days of implementing Kalman filters the computational complexity was a burden. Even with current computers, the computational complexity is still an issue for e.g. large scale systems. Though it is not the purpose here to have a full discussion on Kalman filter implementations, the exercise will lead you experience the consequences of the "devestating results" people dealt with during the Apollo program of trying to save computations. For that purpose, we will consider the following implementation (in pseudo-matlab language):

```
CP = C * P(k|k-1);
Re = CP * C' + R;
iR = inv(Re);
Kk = CP' * iR;
Pi = (P(k|k-1) - Kk * CP);
P(k+1|k) = A * Pi * A' + Q;
```

(for the Ricatti update only).
The system matrices $[A, B, C, D]$, $[Q, R]$ of an example system representing a flexible beam are given in the m-file `Exercise1.m`. At first the parameter `rho` (which represents a small perturbation) is taken equal to 1.

1. Expand the given m-file to implement the suggested implementation of the Kalman filter recursions in **single precision**; thereby only updating the state error covariance matrix $P_{\text{single}}(k|k-1)$.

2. Repeat item 1 in double precision and call the resulted sequence of the state error covariance matrices $P_{\text{double}}(k|k-1)$ (for $k = 1 : N$)). Plot the norm error,

$$
\|P_{\text{single}}(k|k-1) - P_{\text{double}}(k|k-1)\|_2
$$

for $k = 1 : N$.

3. Repeat items 1 and 2 now for a small perturbation, e.g. taken `rho` equal to 1.005 and conclude that the single precision implementation of the Kalman filter **diverges**! (Hereby you may assume the double precision calculations are still "correct")
   Please verify what the spectral radius of the matrices A and $(A - Kk \times C)$ are (in double precision) for $N = 2000$.

[**Hint:** to perform single precision calculations in matlab you have to define the quantities by `single(A)` e.g. for the matrix A.]

The exercise has to provide a copy of the written m-file and the following plot for both values of `rho`.

1. The norm error $\|P_{\text{single}}(k|k-1) - P_{\text{double}}(k|k-1)\|_2$.

2. The reconstruction of the first output with the double precision Kalman filter recursion.

Part 2: Explanation of the Matlab experiment

The numerical representation in the computer differ from the true quantities. Let the numerical representation of the state error covariance matrix $P(k|k-1)$ be denoted by $\overline{P}(k|k-1)$, then its relationship with the true variant is represented as,

$$\overline{P}(k|k-1) = P(k|k-1) + \delta P(k|k-1)$$

1. Develop an error propagation model that describes the recursion of the error $\delta P(k+1|k)$ in terms of the error $\delta P(k|k-1)$ whereby you neglect higher order terms that have an order of magnitude corresponding to that of $\delta P(k|k-1)^2$ or $\delta P(k|k-1)^3$, etc. When the Kalman filter transition matrix is defined as $F(k) = A - AK(k)C$ and $O(\delta^2)$ represent all higher order terms, then show that this error propagation model is given as:

$$\delta P(k+1|k) = F(k)\delta P(k|k-1)F(k)^T + A(\delta P(k|k-1) - \delta P(k|k-1)^T)C^T K(k)^T A^T + O(\delta^2)$$

2. Determine the conditions that the dynamic model derived in item 1 predicts unlimited growth of the error $\delta P(k+1|k)$ for the case `rho` was taken equal to 1.005.

Exercise 2: Estimating an unknown input

Solve exercise 5.3 from [1].

Exercise 3: Inverting Dynamical Systems

Given is a first order dynamic system with unknown input $u(k)$ and measurable output $y(k)$, described as:

$$
\begin{align}
x(k+1) &= ax(k) + bu(k) + w(k) \quad w(k) \sim (0, \sigma_w^2) \tag{1} \\
y(k) &= x(k) + v(k) \quad v(k) \sim (0, \sigma_v^2) \quad E[v(k)w(\ell)] = 0 \quad \forall k, \ell
\end{align}
$$

Then we seek to reconstruct the input for a given recorded output sequence $y^{\text{ref}}(k)$. For that purpose, we assume the input to be modelled by the first order random walk process:

$$u(k+1) = u(k) + w^u(k) \quad w^u(k) \sim (0, \sigma^2) \quad E[w^u(k)\begin{bmatrix} w(\ell) \\ v(k) \end{bmatrix}] = 0 \quad \forall k, \ell$$

For the inversion, we aim at designing a Kalman filter for the following augmented model:

$$
\begin{align}
\begin{bmatrix} x(k+1) \\ u(k+1) \end{bmatrix} &= \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} + \begin{bmatrix} w(k) \\ w^u(k) \end{bmatrix} \tag{2} \\
y(k) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} + v(k)
\end{align}
$$

1. Show that the augmented state space model (2) is observable if $b \neq 0$.

2. For the augmented state space model (2) a stationary Kalman filter is designed. Let the Kalman gain be denoted as K. Derive from this Kalman filter the transfer function from $y^{\text{ref}}(k)$ (or $y(k)$) to the estimated input. Denote this transfer function by $F_u(z)$.

3. Let the input-output behavior of the given first order dynamical system (1) be denoted as,

$$Y(z) = G_u(z)U(z) + H(z)W(z) + V(z)$$

then we can express the reconstructed input as,

$$\hat{U}(z) = F_u(z)G_u(z)U(z) + F_u(z)H(z)W(z) + F_u(z)V(z)$$

Now show that the steady state gain of the transfer function $F_u(z)G_u(z)$ equals 1, i.e.,

$$\lim_{z \to 1} F_u(z)G_u(z) = 1$$

4. Show that based on the result in item 3, the mean value of the reconstructed input $\hat{u}(k)$ in the limit for $k \to \infty$ equals the unkown input value when the latter is constant, i.e, when that constant input is denoted by $\bar{u}$, show that,

$$\lim_{k \to \infty} E[\hat{u}(k)] = \bar{u}$$

5. Verify your results in matlab for $a = 0.9, b = 1, \sigma_w = 0.1, \sigma_v = .01$ by plotting a bode plot of the product $F_u(z)G_u(z)$. And conclude that tracking (in the mean) can be obtained not only in steady state but also for a much wider bandwith.

Exercise 4: Predictor parametrization

Consider the following 2-output, single system:

$$
\begin{aligned}
y_1(k) &= \frac{b_{11}q^{-1} + b_{12}q^{-2}}{(1 + a_1q^{-1})(1 + a_2q^{-1})}u(k) + \frac{1 + c_{11}q^{-1}}{1 + a_1q^{-1}}e_1(k) + \frac{d_{11}q^{-1}}{1 + a_2q^{-1}}e_2(k) \\
y_2(k) &= \frac{b_{21}q^{-1} + b_{22}q^{-2}}{(1 + a_3q^{-1})(1 + a_2q^{-1})}u(k) + \frac{c_{21}q^{-1}}{1 + a_3q^{-1}}e_1(k) + \frac{1 + d_{21}q^{-1}}{1 + a_2q^{-1}}e_2(k)
\end{aligned}
$$

Here $u(k)$ represents the measurable input and $e_1(k), e_2(k)$ are mutually uncorrelated zero mean white noise sequences with unit variance.

1. If we define the vectors $y(k) = \begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix}, e(k) = \begin{bmatrix} e_1(k) \\ e_2(k) \end{bmatrix}$, then determine how the system matrices in the innovation model:

$$
\begin{aligned}
x(k+1) &= Ax(k) + Bu(k) + Ke(k) \\
y(k) &= Cx(k) + e(k)
\end{aligned}
$$

depend on the transfer function parameters a_1, a_2, etc.

2. Check for the values $a_1 = -0.9, a_2 = -0.85, a_3 = -0.95, c_{11} = -.7, c_{21} = -.8, d_{11} = 1, d_{21} = 1; b_{11} = 1, b_{12} = .2, b_{21} = 1, b_{22} = .5$, whether your parametrization is minimal.

3. Show that an innovation model structure of second order exists to paramerize the first (or second) i/o model **only**.

References

[1] M. Verhaegen and V. Verdult, "Filtering and System Identification: A Least Squares Approach", Cambridge University Press, 2007.