HOMEWORK EXERCISE II

LA FOR FILTERING AND IDENTIFICATION

Preferably hand in your solutions as a single PDF file that also includes your m-files. Each student taking part in the exam has to submit one solution set.

Exercise 1: Multiple point data fusion

Given three statistically independent and unbiased estimates $\hat{x}_1, \hat{x}_2$ and $\hat{x}_3$ of the random vector x with the following properties:

$$\begin{aligned} E[\hat{x}_1] &= E[x](:= \overline{x}) \qquad & E[(x-\hat{x}_1)(x-\hat{x}_1)^T] = P_1 > 0 \\ E[\hat{x}_2] &= \overline{x} & E[(x-\hat{x}_2)(x-\hat{x}_2)^T] = P_2 > 0 \\ E[\hat{x}_3] &= \overline{x} & E[(x-\hat{x}_3)(x-\hat{x}_3)^T] = P_3 > 0 \end{aligned}$$

For the exercise it is explicitly stated that the independency means that $E[(x-\hat{x}_i)(x-\hat{x}_j^T)] = 0$ for $i \neq j$. We merge these three estimates into a new estimate $\hat{x}$ by taking the following linear transformation,

$$\hat{x} = M_1\hat{x}_1 + M_2\hat{x}_2 + M_3\hat{x}_3$$

This new estimate has to be unbiased with minimum variance. Show that the covariance matrix $P = E[(x-\hat{x})(x-\hat{x})^T]$ of this 'merged' estimate $\hat{x}$ satisfies,

$$P = \left(\sum_{i=1}^{3} P_i^{-1}\right)^{-1}$$

Exercise 2

Given the set of equations:

$$\begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(N) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} x + \begin{bmatrix} v(1) \\ v(2) \\ \vdots \\ v(N) \end{bmatrix}$$

denoted compactly as,

$$y = Fx + v$$

1. Show that the following least squares solution:

$$\hat{x}_{\text{LS}} = \text{argmin}_x v^T v \ \text{ subject to } \ y = Fx + v$$

   is simply given by,

$$\hat{x}_{\text{LS}} = \frac{1}{N}\sum_{i=1}^{N} y(i)$$

2. When the unknown satisfies $x \sim (0, \sigma_x^2)$ and for the additive perturbation it holds that $v \sim (0, \sigma_v^2 I_N)$, then determine an estimate $\hat{x} = \hat{M}y$, with the vector M given by solving the following minization problem,

$$\hat{M} = \text{argmin}_M E\left[(x - My)(x - My)\right]$$

3. For the conditions given by part 2 of the exercise, show that,

$$E\left[(x-\hat{x}_{\mathrm{LS}})^2\right] > E\left[(x-\hat{x})^2\right]$$

4. Generate a matlab program to verify your theoretical results under the conditions stipulated in part 2 taking $\sigma_x = \sigma_v$. For this it is suggested to perform a monte carlo simulation for fixed $N = 3, 5, 10$ to determine the histograms of the estimates $\hat{x}_{\mathrm{LS}}$ and $\hat{x}$.

Exercise 3: Singular covariance matrices

Consider the stochastic least squares problem with the following 2 sets of information on the unkown random vector $x \in \mathbb{R}^n$:

$$\begin{array}{rcll} \text{Prior:} \quad x & \sim & (\overline{x}, P) & \text{with } \; P > 0 \\ y_1 & = & F_1 x & \quad y_1 \in \mathbb{R}^s \;\; (s < n) \\ y_2 & = & F_2 x + \epsilon & \quad y_2 \in \mathbb{R}^p \quad \text{and } \; \epsilon \sim (0, I) \end{array}$$

with $E[(x-\overline{x})\epsilon^T] = 0$.
Determine an estimate $\tilde{x}$ that combines the given information as:

$$\tilde{x} = M_1 y_1 + M_2 y_2 + N\overline{x}$$

such that it is a Minimum variance unbiased estimate.

Exercise 4: Missing data

A stochastic process $y(k)$ is generated by the following signal generating model (SGM):

$$x(k+1) = Ax(k) + w(k) \quad k \geq 0 \tag{1}$$

$$y(k) = Cx(k) + v(k) \tag{2}$$

with A asymptotically stable and with the zero-mean additive noise sequences $v(k)$ and $w(k)$ satisfying,

$$E\left[\begin{bmatrix} w(k) \\ v(k) \end{bmatrix} \begin{bmatrix} w(\ell)^T v(\ell)^T \end{bmatrix}\right] = \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \Delta(k-\ell) > 0 \quad E\Big[x(0) \begin{bmatrix} w(\ell)^T v(\ell)^T \end{bmatrix}\Big] = 0 \quad \forall \ell$$

Due to errors in the recording system to acquire the data sequence $y(k)$ for $k = 0, \cdots, N$, the sample $y(i)$ $(0 < i < N)$ is **missing**. We look for two ways to find a substitute of this missing value making use of Kalman filtering.
Use a stationary Kalman filter with the samples $y(0), \cdots, y(i-1)$ to produce an estimate of $y(i)$. To use the remaining samples, we determine a backward Kalman filter. For that purpose you need to address the following points:

1. For the forward SGM it holds that it is Markov, i.e. $E[x(k)w(k)^T] = 0$. In a first attempt to formulate a backward model we assume the matrix A is invertible and define the state equation of this inverse model as:

$$x(k) = A^{-1}x(k+1) - A^{-1}w(k) \tag{3}$$

Show that this model violates the so-called backward Markovian property, i.e. $E[x(k+1)w(k)^T] \neq 0$.

2. To get this model (3) backward backward Markovian, a new process noise is $w^b(k)$ is defined as,

$$w^b(k) = w(k) - \hat{M}x(k+1)$$

Determine the matrix $\hat{M}$ such that,

$$\hat{M} = \text{argmin}_M E\Big[(w(k) - Mx(k+1))(w(k) - Mx(k+1))^T\Big]$$

3. Substitute the new process noise $w^b(k)$ into our "first trial" inverse model. Present this model as:

$$x(k) = A_k^b x(k+1) - A^{-1}w^b(k)$$

and show that process noise $w^b(k)$ is white.

4. What is the output equation with this backward state space model to use for backward time prediction to use the part of the data $[y(i+1), \cdots y(N)]$.

References

[1] M. Verhaegen and V. Verdult, "Filtering and System Identification: A Least Squares Approach", Cambridge University Press, 2007.