Design and Implementation of a Test Rig for a Gyro Stabilized Camera System

Examensarbete utfört i Reglerteknik vid Tekniska högskolan i Linköping av

Johannes Eklänge

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The shaker chosen in the test rig is based on a mechanical solution that is described in detail. Additionally all components used in the test rig are described and modelled. The test rig is identified and evaluated from different experiments carried out at PolyTech, where the major part of the identification is based on data collected from accelerometers.

The test rig model is used to develop a controller that controls the frequency and the displacement of the shaker. A three-phase motor is used to control the frequency of the shaker and a linear actuator with a servo is used to control the displacement. The servo controller is designed using observer and state feedback techniques. Additionally, the mount in which the camera system is hanging is modelled and identified, where the identification method is based on nonlinear least squares (NLS) curve fitting technique.

**Keywords**
- shaker
- vibration
- test rig
- accelerometer
- identification
- nonlinear least squares
Abstract

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The shaker chosen in the test rig is based on a mechanical solution that is described in detail. Additionally all components used in the test rig are described and modelled. The test rig is identified and evaluated from different experiments carried out at PolyTech, where the major part of the identification is based on data collected from accelerometers.

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Karl Johannes Eklänge
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Chapter 1

Introduction

PolyTech is a company that manufactures gyro stabilized camera systems. In a gyro stabilized camera system a camera is put inside a gimbal, which is a physical construction that in the general case can rotate around three axes. In PolyTech’s case the gimbal only rotates in two directions. The angular velocity of the inner axis is measured using gyros and momentum motors are put in the gimbal so the camera system can be controlled. The big challenge in developing these systems is to get a stable optical axis i.e., the line which a camera is looking at a scene. In a helicopter application the origin of instability is vibrations and rotations from the helicopter. To get more information of the camera system see [20] in which Peter Skoglar gives a more detailed description of the camera system.

1.1 Problem Description

Vibrations from the helicopter are a major concern when the camera inside the gimbal is stabilized and evaluation of the camera stabilization is an important task. There are different methods to do this evaluation; the easiest way is an ocular evaluation from recorded video images. More accurate methods are based on laser measurements or internal gyro signals of the gimbal. The most natural environment in which to test and evaluate the camera system is during a helicopter flight; there the big benefit is that most flight conditions can be covered. The drawback is that a test flight is time consuming, expensive and hard to monitor, so it is important for PolyTech to have a test rig that simulates the helicopter flight environment.

The camera system has been tested at Sagem in France [6], but it is preferable for PolyTech to have their own test rig, so a test rig is developed that reproduces helicopter vibrations. Both the frequency and the magnitude of the vibrations must be adjustable in the shaker implemented in the test rig. At PolyTech there is a shaker developed that has an adjustable frequency, but it is not possible to control the acceleration. There are some shakers available on the industrial market with an adjustable acceleration, but these shakers are expensive. A more cheaper shaker solution will be described and developed in this master thesis, which is...
Introduction

When the shaker is developed it is important to see what kinds of vibrations are present during a helicopter flight. In Figure 1.1(a) it is shown how the gimbal is mounted to a helicopter. The gimbal is exposed to vibrations origin from the helicopter body, during an test flight in Enköping these vibrations was measured with an accelerometer directed in the $z$-axis defined in Figure 1.1(a). The magnitude of the acceleration at different frequencies are measured using an amplitude spectrum in Figure 1.1(b), the amplitude spectrum is defined in Section 4.4. It can be seen that there are vibrations approximately up to 40 Hz in the range from 0 to 0.2 g.

Using the test rig the mount in which the gimbal is hanging, called the linear mount can be evaluated. It consists of four springs and a mechanical construction that can be described as a roll damper. In this Master Thesis the performance of the mount is investigated by modelling the mount and identifying the model parameters using accelerometers.

1.2 Objectives

The general objectives are:

- Improve an existing test rig for vibration testing of camera systems.
- Validate the new test rig.
- Model the linear mount, in which the gimbal is hanging.
- Identify the model parameters using accelerometers.
1.3 Methods

A big part of this Master Thesis is practical work, since a test rig is actually built. Much time is spent on reading manuals, ordering components, cable wiring and different kind of measurements. The control system and interface of the test rig are programmed in LabVIEW [16], so much time is also spent on learning that language.

A general overview of available shakers on the industrial market is obtained by a literature study. Information about the subject is found on the internet and in technical databases.

The geometry of the shaker solution suggested by Kjell Norén from PolyTech is modelled. This is done by setting up equations for the geometric relations of the shaker. These equations are based on the freedom of the joints and the length of the bars in the shaker. The other components of the test rig are modelled based on knowledge from technical literature and manuals describing sensors and actuators. Model parameters of the test rig are identified from experiments, where accelerometers are used to measure the accelerations of the shaker. In general the parameters are physically derived, so the identification is used to see how well the theory fits reality. Linear regression based on least square estimate (LSE) is used to find linear relations. A more advanced identification method called gray-box identification, a combination of black box identification and physical modelling is used to identify a servo in the test rig. This method is used because it is suitable for identification of parameters in a physical model.

A control system is designed to control the servo. The servo is a single input single output (SISO) system, so the design work is not that difficult. A state feedback and observer based controller is used in the design. By using a pole placement technique to create the control and observer gain the design is reduced to just placing the poles. Compared, to the compensator based controller this controller creates an equivalent feedback and pre-filter compensator in an effective way.

Two accelerometers are used in the identification of the linear mount. One placed above and one under the linear mount. The mount could be identified from velocity and position data integrated from acceleration data. But since it is hard to estimate a position from integrated acceleration data the identification is based on the magnitude of accelerometer data. A curve fitting technique based on the nonlinear least squares (NLS) is used in the identification.

1.4 Thesis Outline

The thesis is divided into the following chapters:

- Chapter 2 contains a description of the shakers available on the industrial market.
- Chapter 3 describes all the components, actuators and sensors used in the test rig.
- Chapter 4 treats all the theory used in this master thesis.
Chapter 5 derives a model of the shaker mechanics. All actuators and sensors from Chapter 3 are also modelled.

Chapter 6 identifies all the model parameters from Chapter 5.

Chapter 7 designs the control system of the test rig.

Chapter 8 evaluates the performance of the test rig.

Chapter 9 derives a model for the linear mount and identifies the model parameters.

Chapter 10 presents all the major results concerning the test rig and the mount. The results are evaluated and suggestions for future work are given.
Chapter 2

Electrical and Mechanical Shakers

In industrial vibration testing applications a shaker is used to produce vibrations. An easy way to get a general view of the equipment used in the industry is to use the industrial search engine Global Spec[3]. The three most important specifications in a shaker actuator are bandwidth, displacement, and power. Naturally, the bandwidth specifies which vibration frequencies the shaker is able to produce. The displacement is the height of the vibrations. The power of the shaker is important since it requires a certain amount of power to shake a mass. As always, different equipment suits applications in different ways. In applications which require large displacements, pneumatics or hydraulics is used. These methods are used for shock testing and not suitable to produce vibrations with high bandwidth. To achieve vibrations with high bandwidth the most accurate and expensive method is to use electromagnetism. The low price alternative to an electromagnetic shaker is a mechanical shaker. Except the high price, the electromagnetic shaker has a lot of benefits over the mechanical shaker. Since the electromagnetic shaker and the mechanical shaker are the two alternatives for the test rig, they will be described in detail below.

2.1 Electromagnetic Shaker

The techniques used in an electromagnetic shaker is described by Ming-Tsau Peng and Tim J. Flack in [15]. An electromagnetic shaker and a loudspeaker have a very similar construction. A moving coil attached to a membrane is producing the sound or vibrations in a loudspeaker. In the same way the vibrations in an electromagnetic shaker is produced by a coil attached to a shaker table. In Figure 2.1(a) the coil here called armature is placed in an electromagnetic flux. The flux is seen in Figure 2.1(b). By driving an AC current in the coil an electromagnetic force perpendicular to the current and the magnetic flux will vibrate the coil. One big benefit with the electromagnetic shaker is the bandwidth which can be
up to several kilo Hertz. It is also easy to generate vibrations with constant accelerations over the frequency spectrum. But in the low-frequency range it is hard to obtain constant acceleration since the displacement will be very large. There are a lot of possibilities when choosing excitations signals in an electromagnetic shaker. For example it is possible to choose multi-sine and white noise as excitation signals. The accuracy and the flexibility of the electromagnetic shaker make it the best shaker on the market.

(a) Cross section of a axisymmetric electromagnetic shaker. (b) Electromagnetic shaker with flux lines.

**Figure 2.1.** Electromagnetic shaker.
2.2 Mechanical Shaker

The origin of a mechanical vibration is always some kind of imbalance. A mechanical shaker uses a motor to produce a rotational motion and through some kind of imbalance a vibration. There are two ways to create this imbalance. The first method based on eccentric masses is almost exclusively used in shakers sold at the industrial market. The other method uses an imbalanced axis called eccentric axis.

2.2.1 Eccentric Mass Shaker

A good description of the eccentric mass shaker is given by Philip Marshall in [14]. The idea is to place masses eccentrically as in Figure 2.2. The motion of the masses is synchronised and counter wise directed. The centrifugal forces of the masses are divided into horizontal and vertical components. The resulting force is pure vertical since the horizontal components cancel each other. This force is sinusoidal with the same frequency as the frequency of the rotational motion. The magnitude of the force is controlled by changing the mass or the eccentricity. The big problem is to control the force during run time. None of all the big vendors of eccentric mass shakers has a system that controls the force without first stopping the shaker. The displacement or the acceleration of the vibrations is dependent of the mass of the object to be shaken and the shaker table.

2.2.2 Eccentric Axis Shaker

In the methods previously described a displacement was obtained by applying a sinusoidal force to a shaker table. With these methods the displacement is dependent on the magnitude of the force and the mass of the shaken system. The principle when vibrations are created with an eccentric axis is a bit different. Here a motion with a given displacement is created thru an eccentric axis. In Figure 2.3 the concept of an eccentric axis is shown. The centre of the lower axis rotates...
around a point that is placed eccentric with the distance $d$ from the axis centre, i.e., the axis rotates around the eccentric point called the eccentric axis. The upper axis must be forced to a vertical motion and will then oscillate with an amplitude $d$.

The principle of the eccentric axis shaker is the same as in a motor when the linear motion of a piston translates into a rotational motion, except here in the shaker application a rotational motion is translated into a linear motion.

The big advantage with this method except that it is much cheaper than the electromagnetic shaker is that the displacement is known. The disadvantage is that the oscillation will be just approximately sine. But with right dimensions on shanks and the eccentricity the approximation is almost perfect.

PolyTech’s old shaker system is an eccentric axis shaker. It has a mechanical adjustable eccentricity, so the displacement is adjustable but unfortunately not during runtime.

Figure 2.3. How a rotational motion translates to a linear motion via an eccentric axis.
Chapter 3

The Test Rig

Figure 3.1. Left: Test rig frame. Right: Shaker mechanics.

In this chapter the mechanical shaker solution suggested by Kjell Norén from PolyTech and all the components used in the test rig are described to get a general view of the test rig functionality. The components are different kind of actuators, sensors and control systems. Actuators used are a linear actuator and a three-phase motor. A built in sensor in the linear actuator measures the stroke length of the linear actuator. For calibration of the test rig accelerometers are used to measure the frequency and the magnitude of the vibrations in the shaker frame. The linear actuator and the three-phase motor are controlled with a servo and a frequency inverter respectively.
3.1 The PolyTech Shaker

The shaker developed in this Master’s Thesis is based on the eccentric axis shaker described in Section 2.2.2, where a rotational motion from a motor is transformed into a linear motion via an eccentric axis. Compared to that shaker, the PolyTech shaker has an adjustable displacement.

Figure 3.2 shows how the components of the shaker are attached to each other. The extension of the motor axis is eccentric and rotates around $B$. A shank is attached to the motor axis. The centre $C$ of the joint between the shank and motor axis will rotate around $B$. A linear actuator is attached to a fixed point $F$ and to a moving carriage in $G$. The frame in which the camera system is attached is forced to a vertical motion and it is attached to a shank in $E$. The mechanical construction with the shanks transforms the rotational motion from the motor to a vertical motion in the frame.

A desired frequency is obtained by driving the motor at a certain speed. The displacement is set by push and pull the carriage forwards and backwards on a rail with the linear actuator. In this way the amplitude of the vibrations is controlled.

Figure 3.2. The PolyTech shaker system in which a camera system is attached to a frame. A motor produces a vertical motion through an eccentric axis. The displacement of the vertical motion is controlled through a mechanical construction and a linear actuator.
3.2 Test Rig Components

The components of the test rig are listed in Table 3.2. In Section 3.2.1 – 3.2.6 the details of the components are discussed briefly.

Table 3.1. Components of the test rig.

| Three-phase motor | Frequency inverter | Linear actuator | Servo | Accelerometer | DAQ-card |

3.2.1 Three-Phase Motor

A three-phase motor from NORD drivsystem AB is used to run the shaker. The power of the motor is 0.55 kW and the motor speed is specified to 1375 rpm\(^1\). This is the speed of the motor when it is powered with a 50 Hz three-phase power source. For this application, the motor power is big enough to get a stable motion. The speed of a three-phase motor is controlled using a frequency inverter.

3.2.2 Frequency Inverter

In the test rig the frequency inverter SK 750/1 from NORD drivsystem AB is used. The mathematics beyond the inverter, not discussed here, is described in the NORDAC vector mc\(^2\) manual [19] as a sensor-less vectorial current control. The inverter produces an optimal three-phase power source with an adjustable frequency so the motor runs with a constant speed. Frequencies up to 100 Hz can be produced by the inverter. The motor speed is proportional to the frequency of the power source.

3.2.3 Linear Actuator

A linear actuator is used to control the position of the small carriage in Figure 3.2. In the test rig the actuator LA12 [11] from LINAK is used. The stroke length of the actuator is 100 mm, so the actuator can perform a linear movement from 0 to 100 mm. The stroke length is measured using a built in potentiometer. When an actuator is chosen for the test rig, the most important specifications of the linear actuator are the power and accuracy, i.e., the speed of actuator is not that crucial. The thrust in both pull and push direction is 750 N. A linear movement is generated by a 24 V DC-motor which is geared to a screw. The spindle pitch of

---

\(^1\)revolutions per minute
\(^2\)mc = motor controller
the actuator is 2 mm and this means that the stroke length of the actuator will increase or decrease 2 mm for each revolution of DC-motor.

3.2.4 Servo

An analogue servo SSA-12/55 [8] from Elmo Motion Control is used to control the DC-motor in the linear actuator. A rotating DC-motor produces an voltage \( V_b \) proportional to the angular velocity \( \omega \) of the motor. This voltage is read and used as feedback for the servo loop. The optimal servo solution is to use a tachometer that measures the angular velocity of the motor. Since this is not the case the accuracy of the servo loop is low. The servo is configured to control a 24 V DC-motor. The servo is controlled using an analogue signal.

3.2.5 Accelerometers

Two 3 axial accelerometers CXL04LP3 [7] from Crossbow Technology is used in the test rig. They are silicon based accelerometers. For high performance, piezo electrical based accelerometers are preferred. But in this application the silicon based accelerometers are good enough. The accelerometers are factory calibrated and the axis has sensibility around 0.5 V/g. The bandwidth of the accelerometer is from DC to 100 Hz.

3.2.6 DAQ-card

A DAQ-card PCI-6221 [17], [18] from National Instruments is used to control the shaker. The DAQ-card and the control algorithm are programmed in National Instruments program LabVIEW [16]. The card measures the analogue signal over the potentiometer and it has two analogue outputs which are used to control the servo and the frequency inverter. In other experiments the card is used to collect data.
Chapter 4

Estimation and Control Theory

In this Chapter, estimation, control and system identification theory is described.

4.1 State Space Models

A very common approach in modelling systems is the state space models. Here, different variants used in this Thesis are defined. In [22] and [23] Glad and Ljung gives a more detailed description of these models. The nonlinear state space model is written as

\[ \dot{x}(t) = f(t, x, u) \]  \hspace{1cm} (4.1a)
\[ y(t) = h(t, x, u) \]  \hspace{1cm} (4.1b)

where, \( u \), \( y \) and the state \( x \) are a \( m \)-dimensional, \( n \)-dimensional and \( p \)-dimensional column vectors respectively. The dimension of the state space model has the order \( n \) i.e., the dimension of the state vector.

**Linear State Space Model**

In a linear state space model the functions \( f \) and \( h \) in (4.1) are written as

\[ f(t, x, u) = Ax(t) + Bu(t) \]  \hspace{1cm} (4.2a)
\[ h(t, x, u) = Cx(t) + Du(t) \]  \hspace{1cm} (4.2b)

so the linear state space model is

\[ \dot{x}(t) = Ax(t) + Bu(t) \]  \hspace{1cm} (4.3a)
\[ y(t) = Cx(t) + Du(t). \]  \hspace{1cm} (4.3b)
Linear State Space Model with Disturbances

Process and measurement disturbances are included in the state space model (4.3) as

\[
\dot{x}(t) = Ax(t) + Bu(t) + Ev_1(t) \quad (4.4a)
\]
\[
y(t) = Cx(t) + Du(t) + v_2(t) \quad (4.4b)
\]

where,

\[
\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}
\]

is white disturbance with intensity

\[
\begin{pmatrix} R_1 & R_{12} \\ R_{12}^T & R_2 \end{pmatrix}. \quad (4.5)
\]

Linear State Space Model on Innovation Form

Disturbances can be included in the linear state space model (4.3) using innovations. The disturbance model is written as an observer, i.e.,

\[
\dot{x}(t) = Ax(t) + Bu(t) + Kv(t) \quad (4.6a)
\]
\[
y(t) = Cx(t) + Du(t) + v(t). \quad (4.6b)
\]

Here the innovation

\[
v(t) = y(t) - Cx(t) - Du(t) \quad (4.7)
\]

is white disturbance with intensity \( R \).

Discrete Time State Space Model

Sampling of systems is described in detail by Ljung in [13]. The continuous state space model (4.6) can be sampled to get a discrete time representation as:

\[
x(t+1) = Fx(t) + Gu(t) + \bar{K}v(t) \quad (4.8a)
\]
\[
y(t) = Cx(t) + Du(t) + v(t) \quad (4.8b)
\]

The matrices \( F \), \( G \) and \( \bar{K} \) is calculated as

\[
F = e^{AT}, \quad G = \int_0^T e^{At}Bdt, \quad \bar{K} \approx \int_0^T e^{At}Kdt, \quad (4.9)
\]

where

\[
 e^{AT} = \mathcal{L}^{-1}(sI - A)^{-1}. \quad (4.10)
\]

In (4.10), \( \mathcal{L}^{-1} \) is the inverse Laplace transform. The approximation of the observer gain \( \bar{K} \) is good for small \( T \). For simple system the discrete time model can be calculated by hand, but with good numerical algorithms the discrete model is easily calculated with computers. Note, with computer tools it is possible to get a better approximation of \( \bar{K} \). In MATLAB (4.8) is calculated using the zero order hold (ZOH) algorithm implemented in the function \texttt{c2d}. 
4.2 Observer and Kalman Filter

Observer and Kalman filter theory is described by Glad and Ljung in [22]. If not all states in a state space model (4.3) are measurable, then an observer is used to calculate the states. Assume that at time \( t = 0 \) the state vector \( x(0) = x_0 \) is known. Ideally, all states will be known in future time by simulating the system:

\[
\dot{x}(t) = Ax(t) + Bu(t) \tag{4.11a}
\]
\[
\dot{x}(0) = x_0 \tag{4.11b}
\]

Of course it is not realistic that the simulated states will fit the real states. To get a perfect match the initial state and the model parameters of the system must be exactly known. From (4.3b) it can be seen that the difference \( y - C\hat{x} - Du \) can be used to evaluate the performance of the simulated state. The simulation (4.11) can be controlled with this difference and the observer is defined as:

\[
\dot{x}(t) = Ax(t) + Bu(t) + K(y(t) - C\hat{x}(t) - Du(t)) \tag{4.12a}
\]
\[
\hat{x}(0) = x_0 \tag{4.12b}
\]

The performance of the observer is determined by the observer gain \( K \), which is the design parameter when designing an observer. If the measurement and process disturbances of a system are known the Kalman filter design is an excellent choice to calculate the observer gain. The Kalman filter is the observer that minimize the prediction error

\[
\hat{x}(t) = x(t) - \hat{x}(t). \tag{4.13}
\]

From the state space model (4.4), the observer gain \( K \) is calculated as

\[
K = (PC^T + NR_{12})R_2^{-1} \tag{4.14}
\]

where, \( P \) is the positive semi definite solution to

\[
AP + PA^T - (PC^T + NR_{12})R_2^{-1}(PC^T + NR_{12})^T + NR_1N^T = 0, \tag{4.15}
\]

which is named the Riccati equation. The solution of (4.15) and the calculation of \( K \) can be done in MATLAB with the command \texttt{lqe}. In cases where a system is modelled in discrete time a Kalman filter in discrete time is used.

4.3 System Identification

System identification is described by Ljung and Glad in [23]. The objective is to estimate the parameters in a parameter vector \( \theta \) of a chosen model structure. The purpose with a model is to make a prediction \( \hat{y}(t|\theta) \) of the value \( y(t) \) where, the prediction error is defined as

\[
\varepsilon(t, \theta) = y(t) - \hat{y}(t|\theta). \tag{4.16}
\]
If the signal $y$ is sampled $N$ times, then the performance of the prediction can be measured with the loss function

$$V_N(\theta) = \frac{1}{N} \sum_{t=1}^{N} e^2(t, \theta). \quad (4.17)$$

The parameters $\theta$ that minimize the loss function (4.17) is chosen as an estimate to $\theta$, i.e.,

$$\hat{\theta} = \min_{\theta} V_N(\theta). \quad (4.18)$$

How well the predicted values $\hat{y}(t)$ fits the measured values $y(t)$ can be described by (4.17). A more developed measurement method is the model fit defined by Ljung in [13] as

$$\text{fit} = 100 \left( 1 - \frac{\sqrt{\sum_{t=1}^{N} (y(t) - \hat{y}(t))^2}}{\sqrt{\sum_{t=1}^{N} (y(t) - \bar{y})^2}} \right) \quad (4.19)$$

where $\bar{y}$ is the mean value of the measured output. It is important to choose the right model structure in identification of systems. The black-box and gray-box model structure and linear regression are discussed here.

### 4.3.1 Black-box Identification

Black-box identification is described by Ljung and Glad in [23] and by Ljung in [13] and is used to describe the relation between the input and the output to a system. This method can be used when there are no interests in describing the physics of a system or when the physics are unknown. In general a discrete time black box structure is written with the discrete time shift operator $q$ as

$$A(q)y(t) = \frac{B(q)}{F(q)}u(t) + \frac{C(q)}{D(q)}e(t) \quad (4.20)$$

where, $A(q)$, $B(q)$, $C(q)$, $D(q)$ and $F(q)$ are polynomials of order $na$, $nb$, $nc$, $nd$ and $nf$ respectively. Different named model structures are obtained by setting some polynomials to unity, these are:

- **ARX**
  
  $$A(q)y(t) = B(q)u(t) + e(t) \quad (4.21)$$

- **OE**
  
  $$y(t) = \frac{B(q)}{F(q)}u(t) + e(t) \quad (4.22)$$

- **ARMAX**
  
  $$A(q)y(t) = B(q)u(t) + C(q)e(t) \quad (4.23)$$
4.3 System Identification

BJ

\begin{equation}
y(t) = \frac{B(q)}{F(q)} u(t) + \frac{C(q)}{D(q)} e(t)
\end{equation}

The big benefit with these model structures is that the model parameters are easily calculated.

In design of control systems there are often a big benefit to have the model structure in a state space form. It is possible to define a black-box state space model as the discrete time state space innovation structure (4.8):

\textbf{State Space}

\begin{align*}
x(t + 1) &= F x(t) + G u(t) + \tilde{K} v(t) \\
y(t) &= C x(t) + D(t) + v(t) \\
x(0) &= x_0
\end{align*}

All parameters of (4.25) including the observer gain \( \tilde{K} \) and the initial state \( x_0 \) are free and in [24] and [12] it is mentioned that the estimation of these parameters can be done using subspace methods.

4.3.2 Gray-box Identification

Rather then having all parameters free as in the black-box state space model structure (4.25), some parameters can be fixed in a gray-box structure if there are insights about the physics in the modelled system. The gray-box model structure \texttt{Idgrey} can be implemented in the Identification Toolbox (\texttt{Sitb}) [12]. The gray-box identification theory is described by the author to \texttt{Sitb}, Prof. Lennart Ljung in [13].

The continuous time model is the most natural representation in gray-box modelling because most physical models are time continuous. However both discrete and continuous state space models are supported in \texttt{Idgrey}.

In two Examples it is shown how a state space model can be parameterized. The background to the Examples is the servo model that will be described in Section 5.4. In Example 4.1 the process disturbance is assumed to be 0 and in Example 4.2 the process disturbance is included in the model description.

\textbf{Example 4.1}

Two parameters \( \theta_1 \) and \( \theta_2 \) affects the dynamics of the system and \( \theta_3 \) the initial state \( x_0 \).

\begin{align*}
\dot{x}(t) &= \begin{pmatrix} 0 & 1 \\ 0 & \theta_1 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ \theta_2 \end{pmatrix} u(t) \\
y(t) &= \begin{pmatrix} 1 & 0 \end{pmatrix} x(t) + v(t) \\
x(0) &= \begin{pmatrix} \theta_3 \\ 0 \end{pmatrix}
\end{align*}
Example 4.2

Kalman filter theory can be used to include process disturbances into the gray-box model structure in an effective way. With known intensity $R_2$ and with $R_{12}^T = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, the gray-box model structure is defined as

\[
\dot{x}(t) = \begin{pmatrix} 0 & 1 \\ 0 & \theta_1 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ \theta_2 \end{pmatrix} u(t) + K(\theta)v(t) \tag{4.27a}
\]

\[
y(t) = \begin{pmatrix} 1 & 0 \end{pmatrix} x(t) + v(t) \tag{4.27b}
\]

\[
x(0) = \begin{pmatrix} \theta_3 \\ 0 \end{pmatrix} \tag{4.27c}
\]

\[
R_1 = \begin{pmatrix} \theta_4 & 0 \\ 0 & \theta_5 \end{pmatrix} \tag{4.27d}
\]

where, $K(\theta)$ is the solution to (4.14) and (4.15).

4.3.3 Linear Regression

Linear regression theory is described by Gustafsson et al. in [9] and by Ljung in [13]. A base

\[
\varphi^T(t) = \begin{pmatrix} \varphi_1(t) & \varphi_2(t) & \cdots & \varphi_n(t) \end{pmatrix}
\]

with linear independent column vectors is used in a linear regression model. A signal $y$ is modelled as the linear combination of the bases in $\varphi^T(t)$ and a parameter vector $\theta$ as,

\[
y(t) = \varphi^T(t)\theta + e, \quad \theta = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_n \end{pmatrix} \tag{4.29}
\]

there $e$ is white noise. The signal $y$ is estimated to

\[
\hat{y}(t) = \varphi^T(t)\theta. \tag{4.30}
\]

If the signal $y$ and the regression vector $\varphi$ are sampled $N$ times, then the matrices

\[
Y = \begin{pmatrix} y(1) \\ y(2) \\ \vdots \\ y(N) \end{pmatrix} \tag{4.31}
\]

and

\[
\Phi = \begin{pmatrix} \varphi^T(1) \\ \varphi^T(2) \\ \vdots \\ \varphi^T(N) \end{pmatrix} \tag{4.32}
\]
4.3 System Identification

are defined, (4.17) can be written as

\[ V_N(\theta) = \frac{1}{N} (Y - \Phi \theta)^T (Y - \Phi \theta). \]  

(4.33)

The pseudo inverse \( \Phi^\dagger \) to \( \Phi \) in an over determined equation system

\[ Y = \Phi \theta \]  

(4.34)

is defined as

\[ \Phi^\dagger = (\Phi^T \Phi)^{-1} \Phi^T. \]  

(4.35)

Using the pseudo inverse, (4.18) is solved as

\[ \hat{\theta} = \Phi^\dagger Y. \]  

(4.36)
4.4 Amplitude and Frequency Estimation

A spectrum based approach to the estimation of the amplitude and frequency of a cosine or sine signal using the discrete Fourier transform (DFT) is treated in this section.

4.4.1 Discrete Time Fourier Transform (DTFT)

A time discrete signal is divided into frequency components using the discrete time Fourier transform (DTFT). Gustafsson et al. [9] gives a good introduction to time discrete transforms. The DTFT for a sampled signal with sample time \( T \)

\[
x[k] = x(kT), \quad k = -\infty, \ldots, +\infty
\]  

is defined as

\[
X_T(e^{i\omega T}) = T \sum_{k=-\infty}^{\infty} x[k]e^{-i\omega kT}. \tag{4.38}
\]

Since a measured signal is finite in time the DTFT must be truncated. The truncated DTFT to the signal

\[
x[n], \quad n = 0, 1, \ldots, N - 1
\]  

is defined as

\[
X^{(N)}_T(e^{i\omega T}) = T \sum_{k=0}^{N-1} x[k]e^{-i\omega kT}. \tag{4.40}
\]

Equations (4.38) and (4.40) can be calculated at an arbitrary frequency \( \omega \) by using the definitions above.

4.4.2 Discrete Fourier Transform (DFT)

The DFT is described by Gustafsson et al. [9] and is defined as

\[
X[n] = \sum_{k=0}^{N-1} x[k]e^{-\frac{2\pi ink}{N}}. \tag{4.41}
\]

The DFT is used to calculate the DTFT at discrete frequency points

\[
\omega = n\omega_0 = \frac{2\pi n}{NT}, \quad n = 0, 1, \ldots, N - 1. \tag{4.42}
\]

With these frequencies chosen it is seen that the DFT is a scaled and sampled version of the DTFT, i.e.,

\[
X[n] = \frac{1}{T}X^{(N)}_T(e^{in\omega_0 T}). \tag{4.43}
\]

\footnote{1In Swedish: tidsdiskret fouriertransform (TDFT)}
4.4 Amplitude and Frequency Estimation

It the TDFT are to be calculated at other frequencies then given by (4.42), then zeros are added to the signal (4.39) that forms the zero padded signal

\[ x_{zp}[m] = \begin{cases} x[m], & m \leq N - 1 \\ 0, & N - 1 < m \leq M - 1 \end{cases}. \]  

(4.44)

The DFT of \( x_{zp} \) is

\[ X_{zp}[m] = \sum_{k=0}^{N-1} x[k] e^{-j2\pi mk/M}. \]  

(4.45)

If the zero padded signal is used to calculate the DTFT it can be calculated at the frequency points

\[ \omega = \frac{2\pi m}{MT}, \quad m = 0, 1, \ldots, M - 1. \]  

(4.46)

In practice, the DFT is implemented using the fast fourier transform (FFT).

4.4.3 Amplitude Spectrum

The amplitudes of a signal at different frequencies can be calculated using an amplitude spectrum. In the tutorial [2] from National Instruments this spectrum is described and defined. For the signal (4.39) the single sided amplitude spectrum is defined as

\[
\text{Amplitude Spectrum (peak)} = \begin{cases} 
\frac{2|X[n]|}{N}, & n = 1, \ldots, N/2 - 1 \\
\frac{|X[n]|}{N}, & n = 0 
\end{cases}. 
\]  

(4.47)

Interpolation of the spectrum can be done using zero padding. If the signal is zero padded with \( M = pN \), then the spectrum is defined as

\[
\text{Amplitude Spectrum (peak)} = \begin{cases} 
\frac{2|X_{zp}[m]|}{N}, & m = p, \ldots, M/2 - 1 \\
\frac{|X_{zp}[m]|}{N}, & m = 0, \ldots, p - 1 
\end{cases}. 
\]  

(4.48)

Note, still after the zero padding the amplitude spectrum is scaled with the factor \( N \).

--- Example 4.3 ---

A \( N = 1024 \) long signal

\[ x[n] = 1 + 0.8 \cos(2\pi 0.25n) + 1.2 \cos(2\pi 0.4n) + e \]  

(4.49)

is sampled with sample time \( T = 1 \) where, \( e \) is Gaussian disturbance with \( \sigma = 0.2 \). The amplitude spectrum of \( x[n] \) is calculated in Figure 4.1(a). Due to leakage the amplitude estimate around 0.4 Hz is not good. In Figure 4.1(b) an amplitude spectrum is calculated using zero padding with \( M = 8N = 8192 \). Good estimates of all amplitudes are obtained.
Estimation and Control Theory

4.5 State Feedback Control

In this Section the state feedback controller is described and how to reconstruct states using observers. It is also shown how to place the poles of the controller and the observer. Considerations of the pole placement are also discussed.

4.5.1 The State Feedback Controller

State space controllers for SISO and MIMO systems is described by Glad and Ljung in [21] and [22] respectively. The theory described here is based on the linear state space representation (4.3) with $D = 0$, as

\begin{align}
\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t).
\end{align}

State space representation of linear system is widely spread and that makes the state feedback controller an attractive approach when designing control systems. Based on the current state $x(t)$ in (4.50), the control signal $u$ is calculated as a linear combination of $x(t)$ as

\begin{equation}
 u(t) = -Lx(t), \quad L = \begin{pmatrix} l_1 & l_2 & \ldots & l_n \end{pmatrix}.
\end{equation}

In servo applications (4.51) is modified by adding the reference signal $r(t)$ as

\begin{equation}
 u(t) = -Lx(t) + L_r r(t).
\end{equation}

By using the controller (4.52) and insert it into the system (4.50), the closed system becomes

\begin{align}
\dot{x}(t) &= (A - BL)x(t) + BL_r r(t) \\
y(t) &= Cx(t).
\end{align}
4.5 State Feedback Control

The dynamics of the controller is described by the eigenvalues of the system matrix $A - BL$. The eigenvalues are the poles of the system. This can be utilized in the design of the controller, by placing the poles of the closed system at arbitrary places if the system is controllable. In [22] Glad and Ljung shows that a system is controllable if the controllability matrix

$$C = \begin{pmatrix} B & AB & \ldots & A^{n-1} B \end{pmatrix}$$

has full rank. By using the Laplace transform the transfer function $G_c(s)$ from $R(s)$ to $Y(s)$ is calculated as

$$Y(s) = G_c(s)R(s) = C(sI - A + BL)^{-1}BL_rR(s).$$

In the design of the controller $L$ and $L_r$ must be chosen so static gain $G_c(0) = 1$. For SISO system a controller is easily calculated that meets this requirement. In the general case all the states are not measurable, so the problem remaining is to calculate the states. This is done using an observer.

4.5.2 Reconstruction of States using an Observer

The observer was defined in Section 4.2. Assume that $D = 0$, then (4.12) becomes:

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + K(y(t) - C\hat{x}(t))$$

$$\hat{x}(0) = x_0$$

(4.56a)

(4.56b)

The reconstruction error $\tilde{x}$ is defined as

$$\tilde{x}(t) = x(t) - \hat{x}(t)$$

(4.57)

and the dynamics of $\tilde{x}$ is calculated as

$$\dot{\tilde{x}}(t) = (A - KC)\tilde{x}(t).$$

(4.58)

How fast the reconstruction error decreases to zero is described by the eigenvalues of the system matrix $A - KC$. The observer gain $K$ is used to place the poles of the observer. If the system (4.3) is observable the poles of the observer can be chosen arbitrary. In [21] Glad and Ljung shows that a system is observable if the observability matrix

$$\mathcal{O} = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix}$$

(4.59)

has full rank.
4.5.3 Pole Placement

For continuous time SISO systems, pole placement techniques are an alternative in the design of control systems. The eigenvalues to the controller system matrix $A - BL$ is calculated using the characteristic equation

$$|\lambda I - (A - BL)| = 0. \quad (4.60)$$

If the system (4.3) is controllable according to (4.54), the poles of $A - BL$ can be placed at $p_1, p_2, \ldots, p_n$. The corresponding characteristic equation of this pole placement becomes

$$(\lambda - p_1) \cdot (\lambda - p_2) \cdot \ldots \cdot (\lambda - p_n) = 0. \quad (4.61)$$

By identification $L$ is calculated using (4.60) and (4.61). Of course there are computer tools available to do these calculations, but for a SISO system with low order this is easily done by hand. The observer poles are placed in the same way as the controller poles.

The eigenvalues of $A - BL$ and $A - KC$ describe the dynamics of the controller and the observer respectively. To get stable systems the poles must be placed in the left half plane. A system is fast if the real part of the poles are placed far from origo. The pole with the real part placed closest to origo is said to be the dominating pole of the system i.e., the dynamic of the system is most dependent on this pole. The drawback with a fast system is that it is more sensitive to both process and measurement noise.

Glad and Ljung suggest in [21] that the poles shall be placed on the bisector in the third and fourth quadrant. They also suggest that the observer shall be a bit faster than the controller. This is intuitively correct since it is obvious that a system must be observed before it can be controlled. In the manual of the MATLAB toolbox Control System Toolbox (CSTB) [1] it is stated that the dynamics of the observer must be faster than the dynamics of the controlled system. The dynamics of the system is found by calculating the eigenvalues to $A$. This is also intuitively correct. If the observer is faster than the system, it will manage to monitor all changes in the system.

4.5.4 Robustness and Sensitivity of the Controller Design

For a controller based on reconstructed states Glad and Ljung calculates in [22] the corresponding feedback compensator $F_y$ as

$$F_y = L(sI - A + BL + KC)^{-1}K. \quad (4.62)$$

The sensitivity function $S$ describes how the output of a control system is effected by process noise and model error. This function is calculated as

$$S = (1 + GF_y)^{-1}. \quad (4.63)$$

The complementary sensitivity function $T$ describes how the output of a control system is effected by measurement noise and how model error effects the system stability. This function is calculated as

$$T = (1 + GF_y)^{-1}GF_y. \quad (4.64)$$
4.6 Nonlinear Least Squares

A nonlinear least squares (NLS) problem can be defined as

$$\min_x g(x) = \min_x \frac{1}{2} (g_1^2(x) + g_2^2(x) + \cdots + g_m(x)^2),$$

(4.65)

where

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$  

(4.66)

A vector valued function is defined as

$$G(x) = \begin{pmatrix} g_1(x) \\ g_2(x) \\ \vdots \\ g_m(x) \end{pmatrix}.$$  

(4.67)

so

$$\min_x g(x) = \frac{1}{2} \|G(x)\|^2_2.$$  

(4.68)

The numerical solution proposed by Lundgren et. al [10] is a second order Taylor approximation around a point $x_k$. That is

$$h(x) = g(x_k) + \nabla_x g(x_k)(x - x_k) + \frac{1}{2}(x - x_k)^T H(x_k)(x - x_k).$$  

(4.69)

The gradient $\nabla_x g(x)$ and the Hessian $H(x)$ are defined as

$$\nabla_x g(x) = \begin{pmatrix} \frac{\partial g}{\partial x_1} \\ \frac{\partial g}{\partial x_2} \\ \vdots \\ \frac{\partial g}{\partial x_n} \end{pmatrix}.$$  

(4.70)

and

$$H(x) = \begin{pmatrix} \frac{\partial^2 g}{\partial x_1^2} & \frac{\partial^2 g}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 g}{\partial x_1 \partial x_n} \\ \frac{\partial^2 g}{\partial x_2 \partial x_1} & \frac{\partial^2 g}{\partial x_2^2} & \cdots & \frac{\partial^2 g}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 g}{\partial x_n \partial x_1} & \frac{\partial^2 g}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 g}{\partial x_n^2} \end{pmatrix}.$$  

(4.71)

The idea is to find an optimum to the approximation by setting the gradient $g(x)$ to 0, that is

$$\nabla_x h(x) = \nabla_x g(x_k) + H(x_k)(x - x_k) = 0.$$  

(4.72)

The search direction $d_k$ is defined as

$$d_k = x - x_k = -H^{-1}(x_k)\nabla_x g(x_k).$$  

(4.73)
When the search direction \( d_k \) is calculated an optimum is found performing a line search in the search direction, i.e
\[
x_{k+1} = x_k + t d_k.
\] (4.74)

Alternatively \( t \) can be set to 1 for a fixed step length. Now the big challenge is to calculate \( H(x) \). The first step is to expand the derivatives in (4.71). One element is expanded as follows
\[
\frac{\partial^2 g}{\partial x_i \partial x_j} = \sum_{l=1}^{m} \frac{\partial}{\partial x_i} \left( \frac{1}{2} \frac{\partial g_l}{\partial x_j} \right) = \sum_{l=1}^{m} \frac{\partial g_l}{\partial x_i} \frac{\partial g_l}{\partial x_j} = \sum_{l=1}^{m} \left( \frac{\partial g_l}{\partial x_i} \frac{\partial g_l}{\partial x_j} + g_l \frac{\partial^2 g_l}{\partial x_i \partial x_j} \right).
\] (4.75)

The Hessian for \( g_l \) is defined as
\[
H_l(x) = \begin{pmatrix}
\frac{\partial^2 g_l}{\partial x_1^2} & \frac{\partial^2 g_l}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 g_l}{\partial x_1 \partial x_n} \\
\frac{\partial^2 g_l}{\partial x_2 \partial x_1} & \frac{\partial^2 g_l}{\partial x_2^2} & \cdots & \frac{\partial^2 g_l}{\partial x_2 \partial x_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial^2 g_l}{\partial x_n \partial x_1} & \frac{\partial^2 g_l}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 g_l}{\partial x_n^2}
\end{pmatrix}
\] (4.76)

and the Jacobian for \( G \) as
\[
J(x) = \begin{pmatrix}
\frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \cdots & \frac{\partial g_1}{\partial x_n} \\
\frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \cdots & \frac{\partial g_2}{\partial x_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial g_m}{\partial x_1} & \frac{\partial g_m}{\partial x_2} & \cdots & \frac{\partial g_m}{\partial x_n}
\end{pmatrix}
\] (4.77)

Note that
\[
J^T(x) J(x) = \sum_{l=1}^{m} \begin{pmatrix}
\frac{\partial g_{l1}}{\partial x_1} & \frac{\partial g_{l1}}{\partial x_2} & \cdots & \frac{\partial g_{l1}}{\partial x_n} \\
\frac{\partial g_{l2}}{\partial x_1} & \frac{\partial g_{l2}}{\partial x_2} & \cdots & \frac{\partial g_{l2}}{\partial x_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial g_{lm}}{\partial x_1} & \frac{\partial g_{lm}}{\partial x_2} & \cdots & \frac{\partial g_{lm}}{\partial x_n}
\end{pmatrix}
\] (4.78)

Now the Hessian \( H(x) \) can be written as
\[
H(x) = J^T(x) J(x) + \sum_{l=1}^{m} g_l(x) H_l(x).
\] (4.79)

The second term in (4.79) is neglected so the Hessian is approximated with the first term. Now \( \nabla_x g(x) \) is left to be calculated. Expanding (4.70) gives that
\[
\nabla_x g(x) = \begin{pmatrix}
\frac{\partial g_1}{\partial x_1} \\
\frac{\partial g_2}{\partial x_2} \\
\vdots \\
\frac{\partial g_n}{\partial x_n}
\end{pmatrix} = \sum_{l=1}^{m} \begin{pmatrix}
\frac{\partial g_{l1}}{\partial x_1} \\
\frac{\partial g_{l2}}{\partial x_2} \\
\vdots \\
\frac{\partial g_{lm}}{\partial x_n}
\end{pmatrix} = J^T G.
\] (4.80)
4.6 Nonlinear Least Squares

Inserting the approximated Hessian and $\nabla_x g(x)$ into (4.72) gives that

$$J(x_k)^T G(x_k) + J(x_k)^T J(x_k) d_k = 0.$$ (4.81)

To calculate the Jacobian $J(x_k)$ numerical differentiation is done around $x_k$. There are two approaches to solve (4.81). In the Gauss-Newton method $d_k$ is solved using least squares techniques in equation

$$J(x_k) d_k = -G(x_k).$$ (4.82)

This method is good but it encounters problem if $J^T J$ is singular. This problem is solved in the Levenberg-Marquardt method where (4.81) is modified as

$$J(x_k)^T G(x_k) + (J(x_k)^T J(x_k) + \lambda_k I) d_k = 0.$$ (4.83)

In [10] Lundgren et al. propose that $\lambda_k$ is chosen bigger than the magnitude of the smallest eigenvalue in $J^T J(x_k)$. Obviously Levenberg-Marquardt is equivalent to Gauss-Newton when $\lambda_k = 0$. It can also be shown that Levenberg-Marquardt tends to the steepest descendant algorithm (not discussed here) when $\lambda_k$ tends to infinity. That is

$$g(x_{k+1}) < g(x_k).$$ (4.84)

4.6.1 Curve Fitting using Nonlinear Least Squares

A nonlinear function dependent on some parameters $x$ is defined as

$$y = \gamma(x, u) + e.$$ (4.85)

Estimate $x$ to fit data can be stated as a nonlinear least squares problem. Suppose output data

$$y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}$$ (4.86)

and input data

$$u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix}$$ (4.87)

are available. The nonlinear least squares approach gives that the functions in (4.67) are defined as

$$g_l(x) = y_l - \gamma_l(x, u_l), \quad l = 1, \ldots, m.$$ (4.88)

The minimization problem

$$\min_x \frac{1}{2} \|G(x)\|_2^2$$ (4.89)

is solved using techniques discussed in Section 4.6.
Chapter 5

Test Rig Modelling

In this Chapter the shaker and other components of the test rig are modelled. There are two main objectives for modelling the test rig. Obviously the model is used to describe the function of the test rig and also the model is used in the design of the controller for the test rig.

5.1 Geometrically Equations of the Shaker

The generation of vibrations and the adjustment of the displacement in the shaker are here described geometrically. The objective is to find the position of the frame as function of the position of the linear actuator and the angular position of the motor. The joints of the shaker was previously defined in Figure 3.2. In Figure 5.1 and 5.2 the shaker is represented with joints and bars. According to these figures,
Figure 5.2. Joint and bar representation of the actuator mounting. The total length of the linear actuator is \( r_a \). The distance \( r_x \) is the distance from the fixed joint \( O \) to the joint \( D \).

the positions of the joints \( A - F \) seen from joint \( O \), are defined as:

\[
\begin{align*}
\mathbf{r}_A &= r_A \cos \alpha \hat{x} + r_A \sin \alpha \hat{y} \\
\mathbf{r}_B &= r_A \hat{x} + \sqrt{r_C^2/A - r_d^2} \hat{y} \\
\mathbf{r}_C &= \mathbf{r}_B + r_d \cos \omega t \hat{x} + r_d \sin \omega t = \\
&= (r_A + r_d \cos \omega t) \hat{x} + \left( \sqrt{r_C^2/A - r_d^2} \right) \hat{y} \\
\mathbf{r}_D &= r_x \cos \alpha \hat{x} + r_x \sin \alpha \hat{y} \\
\mathbf{r}_E &= x_E \hat{x} + y_E \hat{y} \\
\mathbf{r}_F &= x_F \hat{x} \\
\mathbf{r}_G &= (r_x - r_D/G) \cos \alpha \hat{x} + (r_x - r_D/G) \sin \alpha \hat{y}
\end{align*}
\]

(5.1)

The length of bar \( r_{C/A} \), \( r_{E/D} \) and \( r_{G/F} \) can be described with equation

\[
|\mathbf{r}_C - \mathbf{r}_A|^2 = (r_A + r_d \cos \omega t - r_A \cos \alpha) \hat{x} + \left( \sqrt{r_C^2/A - r_d^2} \right) \hat{y}^2 = \\
= (r_A + r_d \cos \omega t - r_A \cos \alpha)^2 + \left( \sqrt{r_C^2/A - r_d^2} \right)^2 = r_C^2/A, \quad (5.2)
\]

\[
|\mathbf{r}_E - \mathbf{r}_D|^2 = (x_E - r_x \cos \alpha) \hat{x} + (y_E - r_x \sin \alpha) \hat{y}^2 = \\
= (x_E - r_x \cos \alpha)^2 + (y_E - r_x \sin \alpha)^2 = r_E^2/D
\]

(5.3)

and

\[
|\mathbf{r}_G - \mathbf{r}_F|^2 = \left[(r_x - r_D/G) \cos \alpha - x_F \right] \hat{x} + (r_x - r_D/G) \sin \alpha \hat{y}^2 = \\
= (r_x - r_D/G)^2 - 2(r_x - r_D/G) x_F \cos \alpha + x_F^2 = r_a^2
\]

(5.4)

respectively. The last equation shows the relationship between \( r_x \) and the actuator length \( r_a \). Solving this equation gives that

\[
r_x = r_D/G + x_F \cos \alpha \pm \sqrt{x_F^2 \cos^2 \alpha - 1 + r_a^2}.
\]

(5.5)
5.1 Geometrically Equations of the Shaker

Since \( r_x \) grows when \( r_a \) grows according to Figure 5.2, the positive solution in (5.5) is chosen, so

\[
r_x = r_D/G + x_F \cos \alpha + \sqrt{x_F^2 (\cos^2 \alpha - 1) + r_a^2}. \tag{5.6}
\]

The position of the frame is found by solving \( y_E \) in (5.3), i.e.,

\[
y_E = r_x \sin \alpha \pm \sqrt{r_{E/D}^2 - (x_E - r_x \cos \alpha)^2}. \tag{5.7}
\]

According to Figure 5.1 it is obvious that the positive solution is to be chosen. The angle \( \alpha \) is calculated using (5.2). To sum up, the position of \( y_E \) which describes the linear oscillating motion of the frame is calculated using the set of equations

\[
\begin{align*}
(r_A + r_d \cos \omega t - r_A \cos \alpha)^2 + \left( \sqrt{r_{C/A}^2 - r_d^2} + r_d \sin \omega t - r_A \sin \alpha \right)^2 &= r_{C/A}^2, \quad (5.8a) \\
r_x &= r_D/G + x_F \cos \alpha + \sqrt{x_F^2 (\cos^2 \alpha - 1) + r_a^2}, \tag{5.8b} \\
y_E &= r_x \sin \alpha + \sqrt{r_{E/D}^2 - (x_E - r_x \cos \alpha)^2}. \tag{5.8c}
\end{align*}
\]

The angle \( \alpha \) in (5.8a) can be solved numerically.

5.1.1 Approximate Solution

Making some assumptions an approximate solution of (5.8) is calculated. This reduces the complexity of the shaker equations. It is assumed that

\[
r_A \gg r_d \quad \tag{5.9}
\]

and

\[
r_{C/A} \gg r_d. \tag{5.10}
\]

In Figure 3.2 it is seen that \( \alpha \) will be small if \( r_A \) is much bigger than \( r_d \). If \( \alpha \) is small it holds that

\[
\cos \alpha \approx 1 \quad \tag{5.11}
\]

and

\[
\sin \alpha \approx \alpha. \tag{5.12}
\]

Note that according to (5.10)

\[
\sqrt{r_{C/A}^2 - r_d^2} \approx r_{C/A}. \tag{5.13}
\]

Using these approximations (5.8a) is written as

\[
r_d^2 \cos^2 \omega t + (r_{C/A} + r_d \sin \omega t - r_A \alpha)^2 = r_{C/A}^2 \tag{5.14}
\]

The solution of \( \alpha \) is

\[
r_A \alpha = r_{C/A} + r_d \sin \omega t \pm \sqrt{r_{C/A}^2 - r_d^2 \cos^2 \omega t} \approx r_{C/A} + r_d \sin \omega t \pm r_{C/A}. \tag{5.15}
\]
Since $\alpha$ oscillates around zero, the solution must be chosen as

$$\alpha = \frac{r_d}{r_A} \sin \omega t. \quad (5.16)$$

The position of the linear actuator described by (5.8b) is approximated as

$$r_x = r_{D/G} + x_F + r_a. \quad (5.17)$$

To simplify the notation, a new parameter $x$ is defined as

$$x \equiv r_{D/G} + x_F + r_a = r_x. \quad (5.18)$$

Finally the oscillation of the frame described by (5.8c) becomes

$$y_E = r_x \alpha + \sqrt{r_F^2/D - (x_E - r_x)^2} =$$

$$= \frac{x}{r_A} r_d \sin \omega t + \sqrt{r_F^2/D - (x_E - x)^2}. \quad (5.19)$$

This leads to the final result that the displacement $s$ of the shaker is

$$s = A(x) \sin \omega t, \quad A(x) = \frac{x}{r_A} r_d. \quad (5.20)$$

From (5.20) a controller to the shaker can be designed. If the distance $x$ can be controlled from 0 to $r_A$, the displacement $s$ of the vibrations is adjustable between 0 and the eccentricity $r_d$. The frequency $\omega$ of the vibrations is dependent on the motor speed.
5.2 Mounting between the Actuator and the Carriage

In the geometrical equations from Section 5.1 it was assumed that the linear actuator lies in line with the joints in O and A when $\alpha = 0$ according to Figure 3.2. Unfortunately in the implementation of the test rig that is not the case. Instead the actuator is placed and mounted according to Figure 5.3. It can be seen in the derivations of the test rig equations below and the evaluation of these equations in Section 8.1, that this mounting will cause problem for small $r_x$. The angle $\beta$ is defined according to Figure 3.2 as

$$\tan \beta = \frac{h}{r_x - b}, \quad (5.21)$$

so

$$r_x = \frac{h}{\tan \beta} + b = \left[ \cot \beta = \frac{1}{\tan \beta} \right] = h \cot \beta + b. \quad (5.22)$$

The geometry of the mounting gives that the position of the joints G and F are

$$r_G = \sqrt{(r_x - b)^2 + h^2} \cos (\alpha - \beta)x + \sin (\alpha - \beta)y = [h > 0] =$$

$$= h \sqrt{\cot^2 \beta + 1} \cos (\alpha - \beta)x + \sin (\alpha - \beta)y \quad (5.23)$$

and

$$r_F = x_F x - h y. \quad (5.24)$$
Because the bar between G and F has the length \( r_a \) it holds that
\[
|r_{G/F}|^2 = |r_G - r_F|^2 = \\
= \left( h \sqrt{\cot^2 \beta + 1 \cos (\alpha - \beta) - x_F} \right)^2 + \\
+ \left( h \sqrt{\cot^2 \beta + 1 \sin (\alpha - \beta) + h} \right)^2 = r_a^2. \quad (5.25)
\]

Now it is possible to generate a new set of equations for the shaker dynamics. The former set of equations (5.8) derived in Section 5.1 will be intact except for (5.8b), this equation is replaced by (5.22) and (5.25), so the new set is
\[
(r_A + r_d \cos \omega t - r_A \cos \alpha)^2 + \left( \sqrt{r_{C/A}^2 - r_d^2 + r_d \sin \omega t - r_A \sin \alpha} \right)^2 = r_{C/A}^2, \quad (5.26a)
\]
\[
\left( h \sqrt{\cot^2 \beta + 1 \cos (\alpha - \beta) - x_F} \right)^2 + \\
+ \left( h \sqrt{\cot^2 \beta + 1 \sin (\alpha - \beta) + h} \right)^2 = r_a^2, \quad (5.26b)
\]
\[
x_a = h \cot \beta + b \quad (5.26c)
\]
and
\[
y_E = r_x \sin \alpha + \sqrt{r_{E/D}^2 - (x_E - x_x \cos \alpha)^2}. \quad (5.26d)
\]

By solving \( \beta \) for an angle \( \alpha \) in (5.26b) numerically the length \( x \) is found using (5.26c). In Section 5.1.1 \( x \) was defined as \( x_x \) when \( \alpha = 0 \). In this case, according to Figure 5.3, the definition of \( x \) is
\[
x \equiv b + x_F + r_a. \quad (5.27)
\]

It is hard to see the performance of the new shaker equations just by looking at them, but it is (5.26b) combined with (5.26c) that causes problem for the shaker dynamics. Note that \( \beta = \pi/2 \) when \( x_x = b \), according (5.26c), so it is no surprise that this will cause strange effects in (5.26b) for small \( x_x \). In Section 8.1 the solution of (5.26) is evaluated.
5.3 Position of the Carriage

The linear actuator is used to control the position of the small carriage on the rail in Figure 3.2. This is done by changing the total length of the actuator $r_a$ in Figure 5.2. The stroke length of the actuator $x_s$ can vary from 0 to its maximum value $x_{s,max}$, i.e.,

$$x_s \in [0, x_{s,max}].$$  \hspace{1cm} (5.28)

The total length of the actuator is defined as

$$r_a = x_s + x_0.$$  \hspace{1cm} (5.29)

By inserting (5.29) into (5.27) the position $x$ which will be used in the servo control of the carriage is calculated as

$$x = b + x_F + r_a = b + x_F + x_s + x_0.$$  \hspace{1cm} (5.30)

The stroke length is measured using a built-in potentiometer. The voltage $y_s$ is measured over the potentiometer. Ideally it holds that

$$x_s = K_s y_s.$$  \hspace{1cm} (5.31)

A bias error is introduced in (5.31) as

$$x_s = K_s y_s + b_s.$$  \hspace{1cm} (5.32)

5.4 Servo Model

The linear actuator is controlled by a servo. In an ideal servo the static velocity $v$ is proportional to the control signal $u$, i.e.,

$$v = ku.$$  \hspace{1cm} (5.33)

The dynamic of the servo is modelled with the time constant $\tau$ as

$$\tau \dot{v} + v = ku.$$  \hspace{1cm} (5.34)
The velocity $v$ is the derivate of the position $x$, so

$$\dot{x} = v.$$  \hfill (5.35)

If the state vector $x$ is defined as

$$x = \left( \begin{array}{c} x \\ v \end{array} \right),$$  \hfill (5.36)

the state space representation of the servo becomes

$$\dot{x} = \left( \begin{array}{cc} 0 & 1 \\ 0 & -\frac{1}{\tau} \end{array} \right) x + \left( \begin{array}{c} 0 \\ \frac{k}{\tau} \end{array} \right) u$$ \hfill (5.37a)

$$y = \left( \begin{array}{cc} 1 & 0 \end{array} \right) x$$ \hfill (5.37b)

and the transfer function

$$Y(s) = G(s)U(s) = \frac{k}{s(\tau s + 1)} U(s).$$ \hfill (5.38)

### 5.5 Frequency of the Frame

The frequency $\omega$ of the oscillating motion of the frame is dependent on the motor speed $v_{\text{rpm}}$. Since the motor speed is measured in revolutions per minute (rpm), the frequency of the frame which is the same as the frequency of the motor axis becomes

$$\omega = K_v v_{\text{rpm}}, \quad K_v = \frac{2\pi}{60} \text{ rad/(s rpm)}.$$ \hfill (5.39)

In Section 3.2.2 it was stated that the motor speed is proportional to the frequency $f_i$ of the three-phase power source given by the inverter. The speed of the motor is specified by the parameter $r$ as

$$v_{\text{rpm}} = r f_i.$$ \hfill (5.40)

The parameter $r$ is calculated using the motor specification where the motor speed $v_{\text{rpm}}$ is specified when $f_i = 50$ Hz. An analogue signal $u_\omega$ is controlling the power source frequency as

$$f_i = K_i u_\omega.$$ \hfill (5.41)

From the control signal $u_\omega$ the frequency $\omega$ is calculated as

$$\omega = r K_i K_v u_\omega = K_\omega u_\omega, \quad K_\omega = r K_i K_v.$$ \hfill (5.42)

A scale and bias error is introduced in (5.42) and modelled as

$$\omega = (1 - a_\omega) K_\omega u_\omega + b_\omega.$$ \hfill (5.43)

The bandwidth of the shaker is calculated by inserting the maximum possible control signal $u_{\omega,max}$ in (5.43) as

$$\omega_{\text{max}} = (1 - a_\omega) K_\omega u_{\omega,max} + b_\omega.$$ \hfill (5.44)
5.6 Acceleration of the Frame in the Approximate Solution

According to (5.20) the displacement of the shaker is

\[ s = A(x) \sin \omega t, \quad A(x) = \frac{x}{r_A} r_d \] (5.45)

This displacement is calculated using the approximate solution from Section 5.1.1, so the real shaker dynamics described in Section 5.2 is not included in (5.45). The acceleration of the frame is calculated by differentiating the displacement twice as

\[ \ddot{s} = -A(x)\omega^2 \sin \omega t. \] (5.46)

The peak of the acceleration is defined as

\[ a = A(x)\omega^2 = \frac{r_d x \omega^2}{r_A}. \] (5.47)

So if the shaker is supposed to vibrate with the frequency \( \omega \) and the acceleration \( a \), the position \( x \) is adjusted to

\[ x = \frac{r_A}{r_d} \frac{a}{\omega^2}. \] (5.48)

This relation is not likely to hold, so it is adjusted with a scale and bias error as

\[ x = (1 - a_s) \frac{r_A}{r_d} \frac{a}{\omega^2} + b_x. \] (5.49)

5.7 Summary

The dynamics of the shaker geometry was modeled in Section 5.1. The approximate solution of this model in Section 5.1.1 showed that it is possible to design a controller that adjust both the displacement and the frequency of the shaker vibrations. Unfortunately a bad implementation of the shaker mechanics caused strange shaker equations, these equations will be further discussed in Section 8.1. The stroke length of the linear actuator is measured with a potentiometer. A servo connected to the DC-motor in the linear actuator is also modelled. Using the potentiometer and the servo, a controller that adjust the stroke length or the vibration displacement is designed. How the frequency inverter controls the speed of the three-phase motor is also modelled. Using this model, the frequency of the shaker vibrations can be controlled. Finally the acceleration of the shaker frame is modelled. The acceleration is dependent on the motor speed and the stroke length. Note, that the approximate solution of the shaker dynamics from Section 5.1.1 was used in the modelling of the acceleration. In fact, the acceleration controller will be designed from the assumption that the acceleration is proportional to the stroke length and the square of the motor speed.
Test Rig Modelling
Chapter 6

Identification of the Test Rig

The parameters used in the modelling of the test rig in Section 5 is identified in this chapter. Some of the parameters are found using technical specifications and manuals. The other parameters are identified using different kinds of experiments. A common used identification method in this chapter, is linear regression described in Section 4.3.3.

6.1 Geometrical Dimensions of the Shaker

Most of the dimensions in the shaker are known from drawings made at PolyTech. Dimensions not derived from drawings are measured using a ruler. In Section 5.2 the dynamics of the shaker is found in (5.26). The parameters in these equations are identified according to Table 6.1.

Table 6.1. Parameter table of the geometrical dimensions in the shaker.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>$r_A$</th>
<th>$r_{C/A}$</th>
<th>$r_{E/D}$</th>
<th>$r_g$</th>
<th>$x_E$</th>
<th>$x_F$</th>
<th>$b$</th>
<th>$h$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value (mm)</td>
<td>207</td>
<td>175</td>
<td>175</td>
<td>9</td>
<td>101</td>
<td>-285</td>
<td>45</td>
<td>17</td>
</tr>
</tbody>
</table>

6.2 Identification of the Carriage Position

In Section 5.3 the position of the moving carriage was modelled in (5.30) and (5.32) as

$$x = b + x_F + x_s + x_0, \quad x_s = K_s y_s + b_s.$$  \hspace{1cm} (6.1)

The values of $b$ and $x_F$ was identified in Section 6.1. In the manual of the linear actuator [11] the length of $x_0$ is found to be 245 mm. In an experiment the stroke length $x_s$ and the voltage $y_s$ over the potentiometer are measured. The stroke length of the linear actuator is measured simply using a ruler. Data from the
Identification of the Test Rig

The stroke length \( x_s \) versus the measured voltage \( y_s \) over the potentiometer is shown in Figure 6.1. The stroke length \( x_s \) can be written as a regression model

\[
y(t) = \varphi^T(t)\theta + e, \tag{6.2}
\]

with

\[
y(t) = x_s, \tag{6.3}
\]
\[
\varphi^T(t) = \begin{pmatrix} y_s & 1 \end{pmatrix} \tag{6.4}
\]

and

\[
\theta = \begin{pmatrix} K_s \\ b_s \end{pmatrix}. \tag{6.5}
\]

The parameters are estimated according Table 6.4.

**Table 6.2.** Parameter table of the stroke length model.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( K_s )</td>
<td>12.1 mm/V</td>
</tr>
<tr>
<td>( b_s )</td>
<td>0.64 mm</td>
</tr>
</tbody>
</table>

6.3 Identification of the Frame Frequency

In Section 5.5 the frequency of the shaker frame was modelled in (5.43) as

\[
\omega = (1 - a_\omega)K_\omega u_\omega + b_\omega, \quad K_\omega = rK_1K_v. \tag{6.6}
\]
6.3 Identification of the Frame Frequency

The frequency of the frame (ω) estimated from acceleration data versus the control signal (u_ω) to the frequency inverter.

The constant $K_v$ was calculated in (5.39) as

$$K_v = \frac{2\pi}{60} \text{ rad/(s rpm)}. \quad (6.7)$$

The parameter $r$ is calculated using the motor specification as

$$r = \frac{1375}{50} \text{ rpm/Hz} = 27.5 \text{ rpm/Hz} \quad (6.8)$$

and the parameter $K_i$ is given from the settings in the frequency inverter as

$$K_i = 5 \text{ Hz/V}, \quad (6.9)$$

hence

$$K_\omega = rK_iK_v = 14.4 \text{ rad/(s V)}. \quad (6.10)$$

In an experiment an accelerometer is attached to the frame of the shaker. The frequency of the frame is estimated from acceleration data using a built in routine in LABVIEW. Using experiment data the scale and bias error in (6.6) are estimated using linear regression as

$$a_\omega = -0.090 \quad (6.11)$$

and

$$b_\omega = 1.66 \text{ rad/s}. \quad (6.12)$$

In Figure 6.2 it can be seen that data fits the model well. The maximum possible value of the control signal $u_\omega$ is 10 V. From (5.44) the bandwidth of the shaker is
calculated as
\[ \omega_{\text{max}} = (1 - a_\omega)K_\omega u_{\omega,\text{max}} + b_\omega = 159 \text{ rad/s}^1. \] (6.13)

Note that the parameter \( K_i \) in (6.9) can be increased to 10 Hz/V. This will increase the bandwidth with a factor 2.

**Table 6.3.** Parameter table of the frame frequency model.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( K_\omega )</td>
<td>14.4 rad/(s rpm)</td>
</tr>
<tr>
<td>( a_\omega )</td>
<td>-0.090</td>
</tr>
<tr>
<td>( b_\omega )</td>
<td>1.66 rad/s</td>
</tr>
<tr>
<td>( \omega_{\text{max}} )</td>
<td>159 rad/s</td>
</tr>
</tbody>
</table>

### 6.4 Identification of the Frame Acceleration

![Figure 6.3](image.png)

**Figure 6.3.** The position \( x \) versus the quotient of the acceleration \( a \) and the square of the frequency \( \omega \).

In Section 5.6 the position \( x \) was modelled as
\[ x = (1 - a_x) \frac{r_A}{r_d} \frac{a}{\omega^2} + b_x. \] (6.14)

The parameters \( r_A \) and \( r_d \) was estimated in Section 6.1. In an experiment an accelerometer is attached to the frame of the shaker. The shaker is adjusted to different frequencies \( \omega \) and different displacements by varying the position \( x \). From acceleration data the peak acceleration \( a \) is estimated using a routine in LabVIEW.

\(^1 159 \text{ rad/s} \) is about 25 Hz
6.4 Identification of the Frame Acceleration

that is able to find the amplitude of a signal in an interesting frequency area. For high frequencies several overtones are present in acceleration data. The peak acceleration is estimated around the interesting frequency $\omega$. In Figure 6.3 data from the experiment is shown. The parameters $a_x$ and $b_x$ in Table 6.4 are estimated using linear regression. The scale error $a_x$ is relatively small, but the bias error $b_x$ is big. The conclusion can be drawn that the acceleration is proportional to the position $x$ and the square of the frequency $\omega$, if the overtones in acceleration data is neglected. In Section 8.1 the shaker equations will be evaluated, then these results will be further discussed.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>$a_x$</th>
<th>$b_x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value</td>
<td>-0.0203</td>
<td>6.75 mm</td>
</tr>
</tbody>
</table>
6.5 Identification of the Servo

To be able to design a controller the servo must be identified. The identification of the servo is made in two steps. The first step is to find the static velocity of the servo loop and the second step is to find the time constant of the servo. Unfortunately it is found that a dead zone is present in the servo loop. A compensation for this dead zone is calculated here.

6.5.1 Identification of the Static Velocity

In (5.33) it was stated that the static velocity \( v \) is proportional to the control signal \( u \) in an ideal servo, as

\[
v = ku. \tag{6.15}
\]

In a series of experiments a constant control signal is applied to the servo. Figure 6.4(a) shows the data from one of these experiments when \( u_x \) is set to 4. The static velocity or the slope is estimated using linear regression. The static velocity was measured and a dead zone was found as seen in Figure 6.4(b). By a transformation from the ideal control signal \( u \) to the real control signal \( u_x \) the nonlinearity is compensated. The function in Figure 6.4(b) is defined as

\[
v = f(u_x) = \begin{cases} 
  K_1 u_x + m_1, & u_x < -\frac{m_1}{K_1} \\
  0, & -\frac{m_1}{K_1} \leq u_x < -\frac{m_4}{K_4} \\
  K_4 u_x + m_4, & -\frac{m_4}{K_4} \leq u_x 
\end{cases}. \tag{6.16}
\]
6.5 Identification of the Servo

(a) Transformation from the ideal control signal \( u \) to the real control signal \( u_x \). Note, the transformation is chosen so \( u_x \) is not smaller or greater than -10 and 10 respectively.

(b) Ideal control signal \( u \) versus the static velocity \( v \). The control signal transformations has made that \( v = ku \) where \( k = 1.41 \).

Figure 6.5. Left: Control signal transformation. Right: Ideal control signal \( u \) versus the static velocity \( v \).

Using (6.15) and (6.16) a transformation from \( u_x \) to \( u \) is calculated as

\[
u = f(u_x) = \begin{cases} \frac{K_1 u_x}{k} + \frac{m_1}{k}, & u_x < -\frac{m_1}{K_1} \\ 0, & -\frac{m_1}{K_1} \leq u_x < -\frac{m_4}{K_4} \\ \frac{K_4 u_x}{k} + \frac{m_4}{k}, & -\frac{m_4}{K_4} \leq u_x \end{cases}
\]

and the transformation from \( u \) to \( u_x \) as

\[
f(u_x) = ku \Rightarrow u_x = f^{-1}(ku) = \begin{cases} \frac{ku-m_1}{K_1}, & u < 0 \\ \frac{ku-m_4}{K_4}, & 0 \leq u \end{cases}
\]

Note that the last transformation has a discontinuity when \( u = 0 \). The parameters in (6.16) are estimated according to Table 6.5. The ideal control signal \( u \) is defined to be in the range from -10 to 10.

\[
u \in [u_{min}, u_{max}] = [-10, 10] \text{ V}
\]

and the real control signal \( u_x \)

\[
u_x \in [u_{x \min}, u_{x \max}] = [-10, 10] \text{ V}
\]

<table>
<thead>
<tr>
<th>Parameter</th>
<th>( K_1 )</th>
<th>( m_1 )</th>
<th>( K_4 )</th>
<th>( m_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value</td>
<td>1.52 mm/(s V)</td>
<td>1.08 mm/s</td>
<td>1.55 mm/(s V)</td>
<td>-1.16 mm/s</td>
</tr>
</tbody>
</table>

Table 6.5. Parameter values of the control signal transformation.
Identification of the Test Rig

After the transformation $u_x$ must not saturate the limitations in (6.20). From (6.18) it is seen that $k$ is chosen as

$$k = \min \left\{ \frac{u_{x \min} K_1 + m_1}{u_{min}}, \frac{u_{x \max} K_4 + m_4}{u_{max}} \right\} = 1.41 \text{ mm/(s V)}. \quad (6.21)$$

### 6.5.2 Gray Box Identification of Servo

![Graphs showing simulated and measured positions](a) Simulated (solid) and measured (x-mark) position. Note, not all measured positions are plotted for readability. There is a good fit between measured and simulated data.

(b) Simulated (solid) and measured (x-mark) position. Note, not all measured positions are plotted for readability. Initial there is a difference between measured and simulated data otherwise it is a relative good fit.

**Figure 6.6.** Identification using the gray-box model structure without disturbances. Left: Negative step response Right: Positive step response.

Parameters in a state space model can be estimated using gray-box identification described in Section 4.3.2. A gray-box routine implemented in the MATLAB System Identification Toolbox (SITB) [12] is used in the identification. In (5.37) the servo was modelled and the parameter $k$ in the model was estimated in (6.21) to $k = 1.41$. If the time constant $\tau$ is included in a parameter as

$$\theta_1 = \frac{1}{\tau}, \quad (6.22)$$

then (5.37) can be written as a gray-box structure:

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 \\ 0 & -\theta_1 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ k\theta_1 \end{pmatrix} u(t) \quad (6.23a)$$

$$y(t) = \begin{pmatrix} 1 & 0 \end{pmatrix} x(t) + v(t) \quad (6.23b)$$

$$x(0) = \begin{pmatrix} \theta_2 \\ 0 \end{pmatrix} \quad (6.23c)$$

This OE² structure does not include process disturbances that might effect the servo. In gray-box identification process disturbances can be modelled in an effec-

²OE = Output Error
6.5 Identification of the Servo


(a) Simulated (solid) and measured (x-mark) position. Note, not all measured positions are plotted for readability. There is a good fit between measured and simulated data.

(b) Simulated (solid) and measured (x-mark) position. Note, not all measured positions are plotted for readability. There is a relative good fit between measured and simulated data.

Figure 6.7. Identification using the gray-box model structure with disturbances. Left: Negative step response Right: Positive step response.

The intensity of the measurement disturbance $v(t)$ is estimated to $R_2 = 0.0065$. It is assumed that there are no cross correlation between the intensities $R_1$ and $R_2$. The gray-box structure with disturbances becomes

\[ x(t) = \begin{bmatrix} 0 & 1 \\ 0 & -\theta_1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ k\theta_1 \end{bmatrix} u(t) + K(\theta)v(t) \]  

\[ y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) + v(t) \]  

\[ x(0) = \begin{bmatrix} \theta_2 \\ 0 \end{bmatrix} \]  

\[ R_1 = \begin{bmatrix} \theta_3 & 0 \\ 0 & \theta_4 \end{bmatrix} \]  

where, the observer gain $K(\theta)$ is calculated using (4.14) and (4.15). Estimation data are collected from two closed loop step response experiments, one negative and one positive step. In Figure 6.6 simulated and measured data is shown where the OE-structure (6.23) is used in the identification. In the negative step response Figure 6.6(a) simulated data fits measured data well, but in Figure 6.6(b) there is a small offset in the initial position and simulated data does not fit measured data as good as it did in the first experiment.

Simulations from the model structure with disturbances (6.24) is shown in Figure 6.7. There are no big differences in the negative step response, but the positive step response has improved.

In Table 6.5.2 the estimated parameters are summarized. The fit defined in (4.19) is also included, which is very good for all experiments and model structures. But it should be stated that the fit is calculated from estimation data and not from...
### Table 6.6. Estimated parameters and fit of the servo models.

<table>
<thead>
<tr>
<th>Experiment</th>
<th>Method</th>
<th>$\theta_1$</th>
<th>$\theta_2$</th>
<th>$\theta_3$</th>
<th>$\theta_4$</th>
<th>fit</th>
<th>$\tau$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Negative</td>
<td>OE</td>
<td>4.40</td>
<td>25.25</td>
<td>—</td>
<td>—</td>
<td>98.3</td>
<td>0.227</td>
</tr>
<tr>
<td>Negative</td>
<td>Disturbance</td>
<td>4.49</td>
<td>25.24</td>
<td>0.626</td>
<td>209.1</td>
<td>99.0</td>
<td>0.223</td>
</tr>
<tr>
<td>Positive</td>
<td>OE</td>
<td>4.72</td>
<td>3.50</td>
<td>—</td>
<td>—</td>
<td>97.9</td>
<td>0.212</td>
</tr>
<tr>
<td>Positive</td>
<td>Disturbance</td>
<td>4.54</td>
<td>3.77</td>
<td>0.690</td>
<td>59.7</td>
<td>98.7</td>
<td>0.220</td>
</tr>
</tbody>
</table>

Validation data. In the disturbance model the intensity $R_1$ differs in the parameter $\theta_4$ between the two experiments, for which the reason is not known. It is interesting to notice that $\theta_3$ is much smaller then $\theta_4$, this is because the first state of the servo model is just an integration of the second state and therefore contains no complex dynamics. When the OE structure is used the time constant $\tau$ differs a bit between the two experiments. The time constant used in future is calculated as the mean of the time constants from the disturbance model, i.e.,

$$
\tau = \frac{0.223 + 0.220}{2} = 0.221.
$$

### 6.5.3 Modification of the Control Signal Transformation

![Graph (a)](image1)

(a) The dead zone is approximated with two square functions.

![Graph (b)](image2)

(b) Transformation from $u$ to $x$.

**Figure 6.8.** Left: Deadzone approximated with a dash dotted and a dotted line in the first and third quadrant respectively. Right: Transformation from $u$ to $u_x$.

The transformation (6.18) from $u$ to $u_x$ has a discontinuity when $u = 0$. This will cause problems in the control system. To avoid the discontinuity the dead zone in $f(u_x)$ is approximated with two square functions according to Figure 6.8(a). The objective is to find a continuous function $g(u_x) \approx f(u_x)$ with a continuous first derivative. If the line and the square function in the third quadrant is defined as

$$
v_1 = K_1 u_x + m_1
$$

(6.26)
6.6 Summary

and

\[ v_2 = K_2 u_x^2 \]  

(6.27)

respectively, the objective is to find an intersection where the first derivate is equal. So the intersection and \( K_2 \) is found as

\[
\begin{align*}
\{ v_1 &= v_2 \\
\dot{v}_1 &= \dot{v}_2 \\
K_1 u_x + m_1 &= K_2 u_x^2 \\
K_1 &= 2K_2 u_x \\
K_2 &= -\frac{2m_1}{K_4} \frac{K_1}{4m_1}.
\end{align*}
\]

(6.28)

The calculation of the intersection in the first quadrant is analogous to the one in (6.28) so an approximate function to \( f(u_x) \) is defined as

\[
f(u_x) \approx g(u_x) = \begin{cases} 
K_1 u_x + m_1, & u_x < -\frac{2m_1}{K_1} \\
K_2 u_x^2, & -\frac{2m_1}{K_1} \leq u_x < 0 \\
K_3 u_x^2, & 0 \leq u_x < -\frac{2m_4}{K_4} \\
K_4 u_x + m_4, & -\frac{2m_4}{K_4} \leq u_x
\end{cases}
\]

(6.29)

Note that the derivate in origo also is continuous. Now the transformation from \( u \) to \( u_x \) in Figure 6.8b is calculated as

\[
\begin{align*}
u_x &= g^{-1}(ku) = \begin{cases} 
k_u - \frac{m_1}{K_1} & u < -\frac{m_1}{k} \\
\sqrt{\frac{ku}{K_2}} - \frac{m_1}{k} & -\frac{m_1}{k} \leq u < 0 \\
\sqrt{\frac{ku}{K_3}} & 0 \leq u < -\frac{m_4}{k} \\
\frac{ku - m_4}{K_4} & -\frac{m_4}{k} \leq u
\end{cases}
\end{align*}
\]

(6.30)

The parameter values in Table 6.5 from the original control signal transformation and the new parameters \( K_2 \) and \( K_3 \) are summarized in Table 6.7. This solution with a transformation must be considered as an improvised solution. The dead zone is quite big, so a new servo would be preferred.

### Table 6.7. Parameter values of the modified control signal transformation.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( K_1 )</td>
<td>1.52 mm/(s V)</td>
</tr>
<tr>
<td>( K_2 )</td>
<td>-0.532 mm/(s V)</td>
</tr>
<tr>
<td>( K_3 )</td>
<td>0.518 mm/(s V)</td>
</tr>
<tr>
<td>( K_4 )</td>
<td>1.55 mm/(s V)</td>
</tr>
<tr>
<td>( m_1 )</td>
<td>1.08 mm/s</td>
</tr>
<tr>
<td>( m_4 )</td>
<td>-1.16 mm/s</td>
</tr>
</tbody>
</table>

6.6 Summary

The geometrical dimensions of the shaker were derived from drawings and measurements using a ruler. The stroke length of the linear actuator was identified
from measurements. From these results the carriage position is calculated using
the potentiometer in the linear actuator.

The identification of the frame frequency from acceleration data turned out well,
which shows that it is possible to control the shaker frequency with high accuracy.
In the identification of the frame acceleration, the acceleration around the chosen
frequency was measured, so the problems with overtones are not taken into account
in the identification. The acceleration is dependent of the carriage position. A big
bias was found between the theoretical and the real position of the carriage, this
is caused by the bad implementation of the carriage mounting, further discussed
in Section 8.1.

The identification of the servo was performed in two steps. First the static velocity
of the servo was measured. A dead zone is present in the servo, so a transformation
from the real to an ideal control signal is made. In a step response experiment the
time constant of the servo was estimated using gray-box identification. Two model
structures without and with disturbance model was tested. The gray-box structure
with disturbance model was found to be the best. The solution with transformed
control signals is very doubtful and must be considered as a temporary solution.
Chapter 7

Control System Design

The tasks of the control system are to control the displacement and the shaker frequency thru the servo and the frequency inverter respectively. On operator level the shaker is adjusted to a desired acceleration and frequency.

7.1 References

The references to the control system is the vibration frequency \( \omega_{\text{ref}} \) and the peak to zero acceleration \( a_{\text{ref}} \). Given these references a reference \( x_{\text{ref}} \) to the position \( x \) is calculated using (5.49), i.e.,

\[
x_{\text{ref}} = K_x \frac{a_{\text{ref}}}{\omega_{\text{ref}}^2} + b_x, \quad K_x = (1 - a_x) \frac{r_A}{r_d}.
\]

(7.1)

For low frequencies high accelerations might be hard to reach because \( x \) is limited. Another problem is that the desired frequency \( f_{\text{ref}} \) cannot be directly sent to the frequency inverter. The problem is that the frequency inverter is much faster than the linear actuator. This can cause vibrations with high frequency and large amplitude which will cause very large accelerations. To prevent this, the frequency reference must be adjusted. All accelerations

\[
a \leq 1.25 a_{\text{ref}}
\]

(7.2)

are accepted. Since the frequency controller is fast it is assumed that the frequency \( \omega \) equals the frequency reference \( \omega_{\text{ref}} \), i.e,

\[
\omega_{\text{ref}} = \omega.
\]

(7.3)

So (7.3) and (5.49) gives that the acceleration \( a \) is calculated as

\[
a = \frac{x - b_x}{K_x} \omega_{\text{ref}}^2.
\]

(7.4)

The inequality (7.2) and (7.4) gives that

\[
\omega_{\text{ref}} \leq \sqrt{\frac{1.25 K_x a_{\text{ref}}}{x - b_x}}.
\]

(7.5)
This leads to the final result that the adjusted frequency reference $\tilde{\omega}_{\text{ref}}$ is chosen as

$$\tilde{\omega}_{\text{ref}} = \min \left( \omega_{\text{ref}}, \sqrt{\frac{1.25K_x a_{\text{ref}}}{x - b_x}} \right). \quad (7.6)$$

### 7.2 Frequency Controller

Controlling the frequency of the shaker frame $\omega$ is an easy task. No feedback information is used in the control of the frequency. It is supposed that the frequency inverter produces correct motor speed. In Section 5.5 the frequency of the shaker frame was modelled in (5.43) as

$$\omega = (1 - a_\omega)K_\omega x + b_\omega, \quad (7.7)$$

so the control signal $u_\omega$ to the inverter shall be chosen as

$$u_\omega = \frac{\omega_{\text{ref}} - b_\omega}{(1 - a_\omega)K_\omega}. \quad (7.8)$$

### 7.3 Servo Controller

The objective in the servo controller design is a controller that is insensible to measurement disturbances, i.e., the bandwidth of the controller is not that important. This problem is solved by placing the poles of the controller to obtain wanted performance. In the design of the control system the servo model (6.23) is used. For convenience it is here repeated

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & -\frac{1}{\tau} \end{pmatrix} x + \begin{pmatrix} 0 \\ \frac{k}{\tau} \end{pmatrix} u \quad (7.9a)$$

$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} x. \quad (7.9b)$$

The parameters $k$ and $\tau$ were estimated in Section 6.5 as

$$k = 1.41 \text{ mm/(s V)} \quad (7.10)$$

and

$$\tau = 0.221. \quad (7.11)$$

### 7.3.1 Design of the Observer

In Section 4.5.3 it was stated that the poles of the observer must be faster than the poles of the controlled system. To avoid having an unnecessary fast observer the poles are chosen just a bit faster than the servo poles. The eigenvalues of the servo are calculated from the characteristic equation as

$$|\lambda I - A| = \left| \lambda I - \begin{pmatrix} 0 & 1 \\ 0 & -\frac{1}{\tau} \end{pmatrix} \right| = \lambda(\lambda + \frac{1}{\tau}) = 0 \quad (7.12)$$
7.3 Servo Controller

hence

\[ \lambda_1 = 0, \quad \lambda_2 = -\frac{1}{\tau} = -4.51. \tag{7.13} \]

It was also stated in Section 4.5.3 that the poles shall be placed on the bisector in the third and fourth quadrant. If the poles are placed in \(-3.5 + 3.5i\) and \(-3.5 - 3.5i\), then the distance to the poles is \(\sqrt{2} \cdot 3.5 \approx 4.95\). This is a bit faster than the fastest pole of the servo, which is placed 4.34 from origo according to (7.13). In Section 4.5.3 it is shown how the observer gain \(K\) is calculated to place the poles correct. The system (7.9) is observable since the observability matrix

\[ O = \begin{pmatrix} C \\ CA \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{7.14} \]

has full rank. The chosen pole placement corresponds to the characteristic equation

\[ (\lambda + 3.5 - 3.5i)(\lambda + 3.5 + 3.5i) = \lambda^2 + 7\lambda + 24.5 = 0. \tag{7.15} \]

The system matrix for the observer is

\[ A - KC = \begin{pmatrix} -k_1 & 1 \\ -k_2 & -\frac{1}{\tau} \end{pmatrix} \tag{7.16} \]

and the eigenvalues of the observer are calculated from the characteristic equation

\[ \left| \lambda - \begin{pmatrix} -k_1 & 1 \\ -k_2 & -\frac{1}{\tau} \end{pmatrix} \right| = \lambda^2 + \left( \frac{1}{\tau} + k_1 \right)\lambda + \frac{k_1}{\tau} + k_2 = 0. \tag{7.17} \]

Using (7.15) and (7.17) the observer gain is calculated as

\[ K = \begin{pmatrix} 2.49 \\ 13.28 \end{pmatrix}. \tag{7.18} \]

7.3.2 Design of the Controller

In Section 4.5.1 the state feedback controller is described. The control signal \(u\) is calculated using (4.52) as

\[ u = -Lx + L_r r, \quad r = x_{ref}, \tag{7.19} \]

where \(r\) is the reference signal \(x_{ref}\) defined in (7.1). From (4.55) the transfer function from \(R(s)\) to \(Y(s)\) is calculated as

\[ Y(s) = C(sI - A + BL)^{-1}L_r R(s) = \frac{\frac{k_1 L_r}{\tau}}{s^2 + \frac{1 + k_2}{\tau} s + \frac{k_1}{\tau}} R(s) \tag{7.20} \]

and since the static gain must be 1 the reference gain \(L_r\) must be set as

\[ L_r = l_1. \tag{7.21} \]
The system (7.9) is controllable since the controllability matrix

\[ C = \begin{pmatrix} B & AB \end{pmatrix} = \begin{pmatrix} 0 & 6.34 \\ 6.34 & -28.7 \end{pmatrix} \]  

has full rank. In Section 4.5.3 it was suggested to choose the observer dynamics a bit faster than the controller dynamics, so the poles of the controller is chosen to \(-3 + 3i\) and \(-3 - 3i\). The corresponding characteristic equation to this pole placement is

\[
(\lambda + 3 - 3i)(\lambda + 3 + 3i) = \lambda^2 + 6\lambda + 18 = 0. \tag{7.23}
\]

The system matrix for the controller is

\[
A - BL = \begin{pmatrix} 0 & -kl_1 \\ \frac{-1}{\tau} & -1 + \frac{kl_2}{\tau} \end{pmatrix} \]  

and the eigenvalues of the observer is calculated from the characteristic equation

\[
\left| \lambda I - \begin{pmatrix} 0 & -kl_1 \\ \frac{-1}{\tau} & -1 + \frac{kl_2}{\tau} \end{pmatrix} \right| = \lambda^2 + \frac{1 + kl_2}{\tau} \lambda + \frac{kl_1}{\tau} = 0. \tag{7.25}
\]

Using (7.23) and (7.25) the feedback controller is calculated as

\[
L = \begin{pmatrix} 2.83 \\ 0.23 \end{pmatrix}. \tag{7.26}
\]

7.3.3 Performance of the Controller in the Frequency Domain

![Graph](image)

(a) Magnitude of the frequency response.  
(b) Phase of the frequency response.

**Figure 7.1.** Frequency response of \(GF_y\).

It is interesting to see how the performance of the chosen pole placement is translated into the frequency domain. A state feedback controller can be described by a compensator based controller with feedback compensator \(F_y\) and pre-filter.
compensator $F_c$. By converting the feedback controller to a compensator based controller the performance in the frequency domain is investigated. In Section 4.5.4 it is shown how to calculate the feedback compensator $F_y$. The frequency response for the open system $G_0 = GF_y$ is shown in Figure 7.1. The crossover frequency $\omega_c$ is estimated to

$$\omega_c = 2.3 \text{ rad.} \quad (7.27)$$

and the phase margin $\Phi_m$ is estimated to

$$\Phi_m = 59^\circ. \quad (7.28)$$

The phase-crossover frequency $\omega_p$ is estimated to

$$\omega_p = 6.7 \text{ rad} \quad (7.29)$$

and the amplitude margin $A_m$ is estimated to

$$A_m = 4.1. \quad (7.30)$$

The phase margin is great so the difference between the peak response and the steady state will be low. The sensitivity function $S$ and the complementary sensitivity function $T$ are calculated as described in Section 4.5.4. In Figure 7.2(a) it can be seen that the performance of the sensitivity function $S$ is acceptable, but the complementary sensitivity function $T$ in Figure 7.2(b) is better. The bandwidth of $T$ is approximately 0.7 rad. This means that the controller is insensible to high frequency measurement disturbances.

![Graphs](image.png)

(a) Sensitivity function $S(i\omega)$. (b) Complementary sensitivity function $T(i\omega)$.

**Figure 7.2.** Left: The sensitivity function $S$. Right: The complementary sensitivity function $T$. 
7.3.4 Discrete Time Implementation of the Controller

How a continuous time state space model is sampled to discrete time state space model was briefly described in Section 4.1. The continuous time control system consist of the observer and the state feedback controller calculated in Section 7.3.1 and Section 7.3.2 respectively:

\[
\dot{x} = (A - KC)\dot{x} + Bu + Ky \tag{7.31a}
\]
\[
u = -L\dot{x} + L_r r \tag{7.31b}
\]

The constants in (7.31) are:

\[
A = \begin{pmatrix} 0 & 1 \\ 0 & -4.51 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 6.36 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 2.49 \\ 13.28 \end{pmatrix}, \quad L = \begin{pmatrix} 2.83 & 0.23 \end{pmatrix}, \quad L_r = 2.83
\]

In most control systems a differential equation is solved to calculate a control signal. The continuous differential equation (7.31a) is solved with a computer by converse (7.31a) to a discrete time differential equation using Zero Order Hold (ZOH) described in the CSTB manual [1]. The sample time \(T\) is set to 0.01 s and the discrete time controller is calculated as:

\[
\hat{x}(t + 1) = \begin{bmatrix} 0.975 & 0.010 \\ -0.128 & 0.955 \end{bmatrix} \hat{x}(t) + \begin{bmatrix} 0.000 \\ 0.062 \end{bmatrix} u(t) + \begin{bmatrix} 0.025 \\ 0.128 \end{bmatrix} y(t) \tag{7.32a}
\]
\[
u(t) = -\begin{bmatrix} 2.83 \\ 0.23 \end{bmatrix} \hat{x}(t) + 2.83r(t) \tag{7.32b}
\]

It can be seen that the discrete time observer (7.32a) has low pass characteristic. From (7.32b) it can be seen that as soon as a new state is calculated the control signal is also updated. The performance of this controller is investigated using measurements in Section 8.2.

7.4 Summary

The models describing the acceleration and frequency of the frame is used to design the control system. The control of the motor speed is an easy task since it is controlled by the frequency inverter. A more difficult task is the design of the servo controller because of the dead zone, discussed in Section 6.5. Anyway the pole placement technique is a good approach to design SISO control systems, since the design is reduced to just placing the poles.
Chapter 8

Evaluation of the Test Rig Performance

8.1 Evaluation of the Shaker Equations

![Graph](image-url)

**Figure 8.1.** The angle $\alpha$ as a function of $\omega t$. The dotted line corresponds to the approximate solution and the numerical solution is the solid line.

The dynamics of the shaker is summarized in the shaker equations (5.26) found in Section 5.2. By solving these equations, conclusions about the shaker performance are drawn. In Section 5.1.1 the displacement for an ideal shaker is found in (5.19). The real shaker is compared with this ideal shaker. The function `fzero` in MATLAB is used to calculate a numerical solution of (5.26a) and (5.26b). The
solution of (5.26a) is a function from the angular position $\omega t$ to the angle $\alpha$ and can be seen as the solid line in Figure 8.1. The mechanical translation from $\omega t$ to $\alpha$ that is independent of the linear actuator length can be seen in Figure 5.1. The dotted line in Figure 8.1 is the approximate solution given by (5.16). To get a better fit between the numerical and the approximate solution, the length $r_{C/A}$ has to be increased. The displacement of the shaker is calculated at different positions $x$. In Figure 8.2 the numerical and approximate displacements are shown. There is only a good fit when $x = 100$ mm. The displacement is greater in Figure 8.2(d) when $x = 1$ mm, than it is in Figure 8.2(c) when $x = 10$ mm. This is no surprise since it was stated in Section 6.4 that the bias error in (5.49) was relatively large. In Figure 8.2 it can be seen that there are strange effects present in the displacement for small $x$. These effects are caused by the bad mounting between the actuator and the carriage.

Figure 8.2. The displacement $y_E$ as a function of $\omega t$ at different positions $x$. The solid line is the numerical solution and the dotted line is the approximate solution.
8.2 Servo Evaluation

The performance of the servo controller (7.32) is investigated in two step response experiments. In the first experiment the reference $x_{\text{ref}}$ is changed from 10 mm to 50 mm, to be able to see how the controller works for big changes in the reference and in the second experiment the reference is changed from 10 mm to 15 mm. The first experiment is shown in Figure 8.3 and it can be seen that the control signal in Figure 8.3(b) saturates at the main part of the step experiment. It should be noted that the position $x$ is the position calculated by the observer (7.32a). Anyway the response in Figure 8.3(a) looks fine. In the second experiment it can be seen in Figure 8.4(a) that $x$ has a oscillating motion when it approaches the reference. Still the performance of the controller is considered as good.

![Figure 8.3](image-url) Left: Step response. Right: Control signal.

![Figure 8.4](image-url) Left: Step response. Right: Control signal.
8.3 Error Analysis of the Acceleration

From (5.49) the acceleration \( a \) is calculated as

\[
a = \frac{x - b_x \frac{r_d}{r_A} \omega^2}{1 - a_x \frac{r_d}{r_A}} = f(x, \omega).
\]  

(8.1)

In [5] Persson and Böiers calculate the error for any multivariable function \( y = f(x) \). The error of the acceleration \( a \) is

\[
|\Delta a| \lesssim \left| \frac{df}{dx} \right| |\Delta x| + \left| \frac{df}{d\omega} \right| |\Delta \omega| = \left| \frac{1}{1 - a_x \frac{r_d}{r_A}} \right| |\Delta x| + \left| \frac{2 x - b_x \frac{r_d}{r_A} \omega}{1 - a_x \frac{r_d}{r_A}} \right| |\Delta \omega|.
\]  

(8.2)

Here \( |\Delta x| \) and \( |\Delta \omega| \) are the control errors. Of course the objective in the design of a control system is to get as good accuracy as possible. Assume that all bias and scale errors are zero, inserting the numerical values of the shaker dimensions into (8.3) gives that

\[
|\Delta a| \lesssim 0.044 \omega^2 |\Delta x| + 0.087 x |\Delta \omega|.
\]  

(8.3)

For large \( x \) it is important that \( |\Delta \omega| \) is small, but the big challenge is to get \( |\Delta x| \) small for high frequencies. This because the upper limit of the acceleration error increases as \( \omega^2 \) according to (8.3).

--- Example 8.1 ---

The wanted upper limit of the acceleration accuracy \( |\Delta a| \) is 0.05g. Assume that \( |\Delta \omega| = 0 \) and \( \omega = 2\pi 20 \text{ rad/s} \), then

\[
|\Delta x| \leq \frac{0.05 \cdot 9.81}{0.044 \cdot 4\pi^2 400} = 0.7 \cdot 10^{-3}.
\]  

(8.4)

This means that \( |\Delta x| \) must not exceed 0.7 mm to obtain the wanted acceleration accuracy.
8.4 Overtones in the Shaker

The performance of the shaker is tested by measuring the acceleration of the frame when different frequencies $\omega$ are applied. The magnitude of the acceleration is calculated using the method based on the DFT described in Section 4.4. Note that this is not the method used in the identification of the frame acceleration in Section 6.4. The acceleration reference $a$ is set to 0.25 g. There are overtones present in the acceleration data collected from the frame. The result from the experiment can be seen in Figure 8.5. Since

$$\sin^2 x = \frac{1 - \cos 2x}{2},$$

the overtones are motivated by the trigonometrically squares present in the shaker equations. In Figure 8.5 it can be seen that the acceleration of the vibrations at $\omega$ differs from the reference acceleration of 0.25 g. For low frequencies it should not be that big difference, but for high frequencies the difference can be motivated by the error analysis discussed in Section 8.3.

![Figure 8.5](image)

**Figure 8.5.** Measured acceleration at different frequencies when the acceleration reference is set to 0.25 g. The optimal result would be a single tone with acceleration 0.25 g. The circles corresponds the controlled frequency. The pluses, the stars and the squares corresponds to the first, second and third overtone respectively.

8.5 Suggested Changes of the Shaker Mechanics

In Section 5.1 a shaker model (5.8) was derived that lead to the approximate model (5.20). In Figure 3.2 and Figure 5.1 the joint $D$ is defined. It is the position of
Table 8.1. Parameter table of the geometrical dimensions in the modified shaker.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>( r_A )</th>
<th>( r_{C/A} )</th>
<th>( r_{E/D} )</th>
<th>( r_{D/G} )</th>
<th>( x_E )</th>
<th>( x_F )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value (mm)</td>
<td>207</td>
<td>175</td>
<td>175</td>
<td>0</td>
<td>9</td>
<td>-285</td>
</tr>
</tbody>
</table>

joint \( D \) that determines the displacement. Ideally the linear actuator is mounted at this joint, i.e., the length \( r_{D/G} = 0 \). With the parameters in (5.8) set according to Table 8.1 and \( x \) set to 1 mm, the numerical solution (5.8) is compared to the approximate solution (5.20). It can be seen in Figure 8.6 that there is a good fit between the numerical and the approximate solution. The deviation could be smaller if the length \( r_{C/A} \) is increased, but still the result is much better compared to the deviation in Figure 8.2(d) where \( x \) is also set to 1 mm. It must be mentioned that this change will reduce the strain on the linear actuator for high frequencies.

Figure 8.6. Oscillation of the modified shaker when \( x \) is set to 1 mm.
Chapter 9

Modelling and Identification of the Linear Mount

The gimbal is attached in a mount consisting of four springs and a mechanical construction that works as a roll damper. When the mount and gimbal is attached to a helicopter, the purpose is to prevent the gimbal from rolling and damp vibrations with high frequency. One important parameter of the mount is the resonance frequency. Knowledge how to manipulate the resonance frequency and the damping is obtained by modelling and identification of the mount.

9.1 Modelling the Linear Mount

For vertical motions the linear mount in which the gimbal is hanging is approximated as a spring and damper system. The four springs in the mount are forced to be equally compressed by the roll damper system, so these springs are replaced with one spring in the model. There are no dampers directly implemented in the mount, but it is assumed that the mount has some naturally damper effects.

9.1.1 Spring and Damper System

In Figure 9.1, $l$ is the static length of the spring under the influence of gravity, i.e., $s$ is the deviation from the static case. The relative velocity acting on the damper is defined as

$$v = v_1 - v_0.$$  \hspace{1cm} (9.1)

From the definitions above the resulting force acting on the mass is found to be

$$F = mg - F_1 - F_2 = -ks - bv$$  \hspace{1cm} (9.2)

Using Newton’s second law, the dynamic for the mass becomes

$$ma_1 = m\ddot{v}_1 = -ks - b(v_1 - v_0).$$  \hspace{1cm} (9.3)
The dynamic for the position is
\[
\dot{s} = v_1 - v_0. \tag{9.4}
\]

If the states
\[
x = \begin{pmatrix} s \\ v_1 \end{pmatrix} \tag{9.5}
\]
and the control signal
\[
u = v_0 \tag{9.6}
\]
are introduced, the spring and damper system from \(v_0\) to \(v_1\) can be written as a state space model
\[
\dot{x} = Ax + Bu, \tag{9.7a}
\]
\[
y = Cx \tag{9.7b}
\]
where
\[
A = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{pmatrix}, \tag{9.8}
\]
\[
B = \begin{pmatrix} -1 \\ \frac{b}{m} \end{pmatrix} \tag{9.9}
\]
and
\[
C = \begin{pmatrix} 0 & 1 \end{pmatrix}. \tag{9.10}
\]
To simplify the notation, \(\frac{k}{m}\) and \(\frac{b}{m}\) are defined as
\[
\beta_1 = \frac{k}{m} \tag{9.11}
\]
and
\[
\beta_2 = \frac{b}{m} \tag{9.12}
\]
9.1 Modelling the Linear Mount

respectively. The transfer function $T(p)$ in

$$y(t) = T(p)u(t)$$  \hspace{1cm} (9.13)

is, according to [22],

$$T(p) = C(pI - A)^{-1}B = \frac{\beta_2 p + \beta_1}{p^2 + \beta_2 p + \beta_1}. \hspace{1cm} (9.14)$$

That is

$$v_1(t) = T(p)v_0(t). \hspace{1cm} (9.15)$$

If the operator $p$ is multiplied into (9.15) and since it holds that

$$pv_i = \dot{v}_i = a_i \hspace{0.25cm} i = 0, 1 \hspace{1cm} (9.16)$$

the dynamic from $a_0(t)$ to $a_1(t)$ becomes

$$a_1(t) = T(p)a_0(t). \hspace{1cm} (9.17)$$

From the transfer function in (9.14) the gain $|T(i\omega)|$ is calculated as

$$|T(i\omega)| = \left| \frac{\beta_1 + i\omega\beta_2}{-\omega^2 + \beta_1 + i\omega\beta_2} \right| = \sqrt{\frac{\beta_1^2 + \omega^2\beta_2^2}{(-\omega^2 + \beta_1)^2 + \omega^2\beta_2^2}}. \hspace{1cm} (9.18)$$
9.2 Identification of the Linear Mount

The spring and damper constants are identified using a curve fitting technique based on the nonlinear least squares treated in Section 4.6.

9.2.1 Curve Fitting using Simulation Data

Equation (9.18) is used to estimate the parameters $\beta_1$ and $\beta_2$ using curve fitting techniques. In reality the gain $|T(i\omega)|$ can be measured using accelerometers. The output and the input to the curve fitting are defined as

$$y = |T(i\omega)| + e = \sqrt{\frac{\beta_1^2 + \omega^2 \beta_2^2}{(-\omega^2 + \frac{\beta_1^2}{\beta_2^2})^2 + \omega^2 \beta_2^2}} + e \quad (9.19)$$

and

$$u = w. \quad (9.20)$$

In the curve fitting manner discussed in Section 4.6.1 the following definitions are done If the parameter vector

$$\mathbf{x} = \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right) = \left( \begin{array}{c} \beta_1 \\ \beta_2 \end{array} \right) \quad (9.21)$$

is defined, then the nonlinear least square problem (4.65) becomes

$$\min_{\mathbf{x}} g(\mathbf{x}) = \min_{\mathbf{x}} \frac{1}{2} \left( g_1^2(\mathbf{x}) + g_2^2(\mathbf{x}) + \cdots + g_m(\mathbf{x})^2 \right), \quad (9.22)$$

where

$$g_l(\mathbf{x}) = y_l - \gamma_l(\mathbf{x}, u_l) = y_l - \sqrt{\frac{x_1^2 + u_l^2 x_2^2}{(-u_l^2 + x_1)^2 + u_l^2 x_2^2}}, \quad l = 1, \ldots, m. \quad (9.23)$$

To generate the signals $y$ and $u$, the state space model described in (9.7) — (9.10) is implemented in Simulink. The parameters $\beta_1$ and $\beta_2$ are set to 3000 and 20 respectively. The gain in (9.18) is calculated at different frequencies and in reality an arbitrary technique might be chosen. In simulation a single sine signal is chosen as input. In Figure 9.3(a) the gain from different simulations is shown. When estimating the parameters $\beta_1$ and $\beta_2$ it is important to get a good initial guess. In Figure 9.3(a) it can be seen that the resonance frequency $\omega_r$ is approximately

$$\omega_r \approx 50 \text{ rad/s}. \quad (9.24)$$
9.2 Identification of the Linear Mount

Table 9.1. Estimated parameters from simulation data.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>True value</th>
<th>Estimate1 (noise σ = 0.1)</th>
<th>Estimate2</th>
</tr>
</thead>
<tbody>
<tr>
<td>β₁</td>
<td>3000</td>
<td>2999.7</td>
<td>3026.1</td>
</tr>
<tr>
<td>β₂</td>
<td>20</td>
<td>19.998</td>
<td>19.820</td>
</tr>
</tbody>
</table>

Assume that the system doesn’t contain any damping, i.e. β₂ = 0. From (9.18) it can be seen that a good initial guess of β₁ is

\[ β_{1\text{init}} = \omega_r^n = 2500. \] (9.25)

The initial value of β₂ is set to a value greater than 0, that is

\[ β_{2\text{init}} = 1. \] (9.26)

The parameters are estimated according to the fourth column in Table 9.1. The minimization is performed using the nonlinear least squares over the norm in Figure 9.3(b).

![Figure 9.3.](image)
9.3 Curve Fitting using Measured Data

![Diagram](image)

(a) Accelerometers are attached to the frame and the gimbal body.
(b) Curve fitted to measured data. Only data points below 55 rad/s is used in the identification.

**Figure 9.4.** Left: Experiment setup. Right: Curve fit using measured data.

Several static vibration tests were performed on the mount. An accelerometer was attached to the frame of the shaker and one on the gimbal body according to Figure 9.4(a). The measurements did not turn out well. First of all, a small but ocular observable rotational motion of the gimbal body was observed. The gimbal side on which the accelerometer was attached seemed to vibrate a bit more than the opposite side. The gimbal is made of composite, so for high frequencies, resonances can cause the structure of the gimbal to vibrate in an unexpected way. It would have been preferable to put an accelerometer inside the gimbal. Theoretically the linear mount should damp vibrations with high frequency, but this damping was hard to measure.

By examining the FFT’s of the accelerometer signals the gain $|T(i\omega)|$ was estimated according to Figure 9.4(b). The ocular estimation during the experiment is that the resonance frequency lies around 50 rad/s. High frequency data points above 55 rad/s has been removed in the identification. In Figure 9.4(b) a curve is fitted to data. The parameters $\beta_1$ and $\beta_2$ are estimated according to Table 9.2. Because of the bad measurements this result should be used with caution. In theory the identification method based on the nonlinear least squares curve fitting technique works well, which is shown using simulation data. The problems in this case with the identification of the linear mount were either measurement technical or that the model of the linear mount did not describe the reality in a good way.

**Table 9.2.**

<table>
<thead>
<tr>
<th>$\beta_1$</th>
<th>$\beta_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2550</td>
<td>25.3</td>
</tr>
</tbody>
</table>
Chapter 10

Conclusions and Future Work

In this Master Thesis a test rig was designed and developed for vibration testing of gyro stabilized camera systems and the mount in which the camera system is hanging was tested in the test rig. Working with this thesis gave much knowledge in industrial development.

10.1 Conclusions about the Test Rig

Ideally it is possible to design a mechanical shaker with an adjustable displacement according to the approximate shaker equations in Section 5.1.1. Unfortunately the mechanics of the implemented shaker caused overtones for vibrations with high frequency, but it is shown in Section 5.1.1 that a change in the mechanics will increase the performance of the shaker dramatically. Hopefully all overtones will vanish if this change is made.

A control system was developed that controls the frequency and the acceleration of the shaker frame. The control of the acceleration was based on the fact that the acceleration is proportional to the square of the frequency and the position of the linear actuator. For high frequencies it is hard to control the acceleration, which is shown in the error analysis in Section 8.3.

The linear actuator in the test rig was chosen to be as strong and accurate as possible, to make sure to get an accurate positioning. For the control of the linear actuator an analogue servo was used, unfortunately a dead zone caused problem in the design of the servo controller. This problem was solved with a control signal transformation, which might seem to be a strange solution. The servo controller was designed as a state feedback controller, since the linear actuator was not equipped with a tachometer the states were reconstructed using an observer. Pole placement technique gave an accurate and relatively fast controller that is also insensible to measurement disturbances.

The frequency inverter is able to control the shaker frequency with high accuracy,
which could be seen in the calibration of the inverter in Section 6.3. When this calibration was made there were no camera system attached in the shaker frame, so there is a possibility that the inverter is sensitive to load changes. To make sure that the shaker is insensible to load changes, the frequency inverter must be configured to control the motor speed with feedback information from a sensor that measures the speed of the motor shaft.

The bandwidth of the shaker is 25 Hz, but it can be increased up to 50 Hz. Since the accuracy of the acceleration controller is low for high frequencies at this moment, the bandwidth can not likely be increased. Still the bandwidth needs to be increased to be able to produce all vibrations that originate from a helicopter.

10.2 Conclusions about the Linear Mount

Curve fitting based on the nonlinear least squares was treated in Section 4.6. An identification method based on this technique was used to identify the spring and damper constants in a spring-damper system by measure the gain between two accelerometers. This method works very well on simulation data, which verifies that this method is usable. In an experiment the gain between two accelerometers was measured. Due to resonances in the fixture where one of the accelerometer was attached this experiment did not turn out that well, but with a better experiment setup this method might give good results.

10.3 Future Work

If the mechanics is changed according to Section 8.5 the shaker performance will increase and there is high hopes that the overtones will vanish. The bandwidth of the shaker can be increased, if the new mechanics gives more accurate accelerations for high frequencies. If the shaker produces a pure sine signal, it might be possible to use accelerometers to control the shaker acceleration, as it is done in some electromagnetic shakers. Aarts describes a fast recursive algorithm in [4] for estimation of both the amplitude and frequency of single tone signal, which could be used in a more advanced control system.

A new servo must be implemented to get rid of the control signal transformation, so a digital servo that is easier to configure would help a lot.

A sensor connected to the frequency inverter that measures the motor speed would make sure that right frequencies are generated by the shaker.

To get a more secure and stable implementation of the control system, it should be implemented on an imbedded system, for example a PC104.
Bibliography


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