Extended Target Tracking with a Cardinalized Probability Hypothesis Density Filter

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Abstract

This technical report presents a cardinalized probability hypothesis density (CPHD) filter for extended targets that can result in multiple measurements at each scan. The probability hypothesis density (PHD) filter for such targets has already been derived by Mahler and a Gaussian mixture implementation has been proposed recently. This work relaxes the Poisson assumptions of the extended target PHD filter in target and measurement numbers to achieve better estimation performance. A Gaussian mixture implementation is described. The early results using real data from a laser sensor confirm that the sensitivity of the number of targets in the extended target PHD filter can be avoided with the added flexibility of the extended target CPHD filter.

Keywords: Multiple target tracking, extended targets, random sets, probability hypothesis density, cardinalized, PHD, CPHD, Gaussian mixture, laser.

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***Abstract*—This report presents a cardinalized probability hypothesis density (CPHD) filter for extended targets that can result in multiple measurements at each scan. The probability hypothesis density (PHD) filter for such targets has already been derived by Mahler and a Gaussian mixture implementation has been proposed recently. This work relaxes the Poisson assumptions of the extended target PHD filter in target and measurement numbers to achieve better estimation performance. A Gaussian mixture implementation is described. The early results using real data from a laser sensor confirm that the sensitivity of the number of targets in the extended target PHD filter can be avoided with the added flexibility of the extended target CPHD filter.**

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I. Introduction

The purpose of multi target tracking is to detect, track and identify targets from sequences of noisy, possibly cluttered, measurements. The problem is further complicated by the fact that a target may not give rise to a measurement at each time step. In most applications, it is assumed that each target produces at most one measurement per time step. This is true for some cases, e.g. in radar applications when the distance between the target and the sensor is large. In other cases however, the distance between target and sensor, or the size of the target, may be such that multiple resolution cells of the sensor are occupied by the target. This is the case with e.g. image sensors. Targets that potentially give rise to more than one measurement are denoted as extended.

Gilholm and Salmond [1] presented an approach for tracking extended targets under the assumption that the number of recieved target measurements in each time step is Poisson distributed. They show an example where they track point targets which may generate more than one measurement, and an example where they track objects that have a 1-D extension (infinitely thin stick of length l). In [2] a measurement model was suggested which is an inhomogeneous Poisson point process. At each time step, a Poisson distributed random number of measurements are generated, distributed around the target. This measurement model can be understood to imply that the extended target is sufficiently far away from the sensor for its measurements to resemble a cluster of points, rather than a geometrically structured ensemble. A similar approach is taken in [3], where the so-called track-before-detect theory is used to track a point target with a 1-D extent.

Random finite set statistics (FISST) proposed by Mahler has become a rigorous framework for target tracking adding the Bayesian filters named as the probability hypothesis density (PHD) filter [4] into the toolbox of tracking engineers. In a PHD filter the targets and measurements are modeled as finite random sets, which allows the problem of estimating multiple targets in clutter and uncertain associations to be cast in a Bayesian filtering framework [4]. A convenient implementation of a linear Gaussian PHD-filter was presented in [5] where a PHD is approximated with a mixture of Gaussian density functions. In the recent work [6], Mahler gave an extension of the PHD filter to also handle extended targets of the type presented in [2]. A Gaussian mixture implementation for this extended target PHD filter was presented in [7].

In this report, we extend the works in [6] and [7] by presenting a cardinalized PHD (CPHD) filter [8] for extended target tracking. To the best of the authors' knowledge and based on a personal correspondence [9], no generalization of Mahler's work [6] has been made to derive a CPHD filter for the extended targets of [2]. In addition to the derivation, we also present a Gaussian mixture implementation for the derived CPHD filter which we call extended target (tracking) CPHD (ETT-CPHD) filter. Further, early results on laser data are shown which illustrates robust characteristics of the ETT-CPHD filter compared to its PHD version (called naturally as ETT-PHD).

This work is a continuation of the first author's initial derivation given in [10] which was also implemented by the authors of this study using Gaussian mixtures and discovered to be extremely inefficient. Even though the formulas in [10] are correct (though highly inefficient), the resulting Gaussian mixture implementation also causes problems with PHD coefficients that can turn out to be negative. The resulting PHD was surprisingly still valid due to the identical PHD components whose weights were always summing up to their true positive values. The derivation presented in this work gives much more efficient formulas than [10] and the resulting Gaussian mixture implementation works without any problems.

The outline for the remaining parts of the report is as follows. We give a brief description of the problem in Section II where we define the related quantities for the derivation of the ETT-CPHD filter in Section III. Note here that we are unable to supply an introduction in this work to all the details of the random finite set statistics for space considerations. Therefore, Section II and especially Section III require some familiarity with the basics of the random finite set statistics. The unfamiliar reader can consult Chapters 11 (multitarget calculus), 14 (multitarget Bayesian filter) and 16 (PHD and CPHD filters) of [11] for an excellent introduction. Section IV describes the Gaussian mixture implementation of the derived CPHD filter. Experimental results based on laser data are presented in Section V with comparisons to the PHD filter for extended targets. Section VI contains conclusions and thoughts on future work.

II. Problem Formulation

PHD filter for extended targets has the standard PHD filter update in the prediction step. Similarly, ETT-CPHD filter would have the standard CPHD update formulas in the prediction step. For this reason, in the subsequent parts of this report, we restrict ourselves to the (measurement) update of the ETT-CPHD filter. We consider the following multiple extended target tracking update step formulation for the ETT-CPHD filter.

- We model the multitarget state X_k as a random finite set $X_k = \{x_k^1, x_k^2, \dots, x_k^{N_k^T}\}$ where both the states $x_k^j \in \mathbb{R}^{n_x}$ and the number of targets N_k^T are unknown and random.
- The set of extended target measurements, $Z_k = \{z_k^1, \dots, z_k^{N_k^z}\}$ where $z_k^i \in \mathbb{R}^{n_z}$ for $i = 1, \dots, N_k^z$, is distributed according to an independent identically distributed (i.i.d.) cluster process. The corresponding set likelihood is given as

$$f(Z_k|x) = N_k^z! P_z(N_k^z|x) \prod_{z_k \in Z_k} p_z(z_k|x) \tag{1}$$

  where $P_z(\,\cdot\,|x)$ and $p_z(\,\cdot\,|x)$ denote the probability mass function for the cardinality N_k^z of the measurement set Z_k given the state $x \in \mathbb{R}^{n_x}$ of the target and the likelihood of a single measurement. Note here our convention of showing the dimensionless probabilities with "P" and the likelihoods with "p".
- The target detection is modeled with probability of detection $P_D(\,\cdot\,)$ which is a function of the target state $x \in \mathbb{R}^{n_x}$.
- The set of false alarms collected at time k are shown with $Z_k^{FA} = \{z_k^{1,FA}, \dots, z_k^{N_k^{FA},FA}\}$ where $z_k^{i,FA} \in \mathbb{R}^{n_z}$ are distributed according to an i.i.d. cluster process with the set likelihood

$$f(Z_{FA}) = N_z^{FA}! P_{FA}(N_k^{FA}) \prod_{z_k \in Z_k^{FA}} p_{FA}(z_k) \tag{2}$$

  where $P_{FA}(\,\cdot\,)$ and $p_{FA}(\,\cdot\,)$ denote the probability mass function for the cardinality N_k^{FA} of the false alarm set Z_k^{FA} and the likelihood of a single false alarm.
- Finally, the multitarget prior $f(X_k|Z_{0:k-1})$ at each estimation step is assumed to be an i.i.d. cluster process.

$$\begin{aligned} f(X_k|Z_{0:k-1}) =& N_{k|k-1}! P_{k+1|k}(N_{k+1|k}) \\ &\times \prod_{x_k \in X_k} p_{k+1|k}(x_k) \end{aligned} \tag{3}$$

  where

$$p_{k|k-1}(x_k) \triangleq N_{k|k-1}^{-1} D_{k|k-1}(x_k) \tag{4}$$

  with $N_{k|k-1} \triangleq \int D_{k|k-1}(x_k)\, \mathrm{d}x_k$ and $D_{k|k-1}(\,\cdot\,)$ is the predicted PHD of X_k.

Given the above, the aim of the update step of the ETT-CPHD filter is to find the posterior PHD $D_{k|k}(\,\cdot\,)$ and the posterior cardinality distribution $P_{k|k}(\,\cdot\,)$ of target finite set X_k.

III. CPHD filter for Extended Targets

The probability generating functional (p.g.fl.) corresponding to the updated multitarget density $f(X_k|Z_{0:k})$ is given as

$$G_{k|k}[h] = \frac{\frac{\delta}{\delta Z_k} F[0,h]}{\frac{\delta}{\delta Z_k} F[0,1]} \tag{5}$$

where

$$F[g,h] \triangleq \int h^X G[g|X] f(X|Z_{k-1}) \delta X, \tag{6}$$

$$G[g|x] \triangleq \int g^Z f(Z|X) \delta Z \tag{7}$$

with the notation h^X showing $\prod_{x\in X} h(x)$. The updated PHD $D_{k|k}(\,\cdot\,)$ and the updated probability generating function $G_{k|k}(\,\cdot\,)$ for the number of targets are then provided with the identities

$$D_{k|k}(x) = \frac{\delta}{\delta x} G_{k|k}[1] = \frac{\frac{\delta}{\delta x}\frac{\delta}{\delta Z_k} F[0,1]}{\frac{\delta}{\delta Z_k} F[0,1]}, \tag{8}$$

$$G_{k|k}(x) = G_{k|k}[x] = \frac{\frac{\delta}{\delta Z_k} F[0,x]}{\frac{\delta}{\delta Z_k} F[0,1]}. \tag{9}$$

In the equations above, the notations $\frac{\delta\,\cdot}{\delta X}$ and $\int \cdot\, \delta X$ denote the functional (set) derivative and the set integral respectively.

- Calculation of $G[g|X]$: The p.g.fl. for the set of measurements belonging to a single target with state x is as follows.

$$G_x[g|x] = 1 - P_D(x) + P_D(x) G_z[g|x] \tag{10}$$

  where $G_z[g|x]$ is

$$G_z[g|x] = \int g^X f(Z|x) \delta Z. \tag{11}$$

  Suppose that, given the target states X, the measurement sets corresponding to different targets are independent. Then, we can see that the p.g.fl. for the measurements belonging to all targets becomes

$$G_X[g|X] \triangleq (1 - P_D(x) + P_D(x) G_z[g|x])^X. \tag{12}$$

With the addition of false alarms, we have

$$G[g|X] = G_{FA}[g](1 - P_D(x) + P_D(x)G_z[g|x])^X. \tag{13}$$

- Calculation of $F[g,h]$: Substituting $G[g|X]$ above into the definition of $F[g,h]$ we get

$$\begin{aligned} F[g,h] =& \int h^X G_{FA}[g](1 - P_D(x) + P_D(x)G_z[g|x])^X \\ & \times f_{k|k-1}(X|Z_{k-1})\delta X & (14) \\ =& G_{FA}[g] \int (h(1 - P_D(x) + P_D(x)G_z[g|x]))^X \\ & \times f_{k|k-1}(X|Z_{k-1})\delta X & (15) \\ =& G_{FA}[g]G_{k|k-1}\big[h(1 - P_D(x) + P_D(x)G_z[g|x]\big] & (16) \\ =& G_{FA}[g]G_{k|k-1}\big[h(1 - P_D + P_DG_z[g]\big] & (17) \end{aligned}$$

where we omitted the arguments x of the functions $P_D(\cdot)$ for simplicity. For a general i.i.d. cluster process, we know that

$$G[g] = G(p[g]) \tag{18}$$

where $G(\cdot)$ is the probability generating function for the cardinality of the cluster process; $p(\cdot)$ is the density of the elements of the process and the notation $p[g]$ denotes the integral $\int p(x)g(x)\,\mathrm{d}x$. Hence,

$$\begin{aligned} F[g,h] =& G_{FA}(p_{FA}[g]) \\ & \times G_{k|k-1}\Big(p_{k|k-1}\big[h(1 - P_D + P_DG_z(p_z[g]))\big]\Big). \end{aligned} \tag{19}$$

We first present the following result.

Theorem 3.1 (Derivative of the Prior p.g.fl.): The prior p.g.fl. $G_{k|k-1}\big[h(1 - P_D + P_DG_z[g]\big]$ has the following derivatives.

$$\begin{aligned} &\frac{\delta}{\delta Z}G_{k|k-1}\Big(p_{k|k-1}\big[h(1 - P_D + P_DG_z(p_z[g]))\big]\Big) \\ &= G_{k|k-1}\Big(p_{k|k-1}\big[h(1 - P_D + P_DG_z(p_z[g]))\big]\Big)\delta_{Z=\phi} \\ &\quad + \sum_{\mathcal{P}\angle Z} G_{k|k-1}^{(|\mathcal{P}|)}\Big(p_{k|k-1}\big[h(1 - P_D + P_DG_z(p_z[g]))\big]\Big) \\ &\quad \times \prod_{W\in\mathcal{P}} p_{k|k-1}\Big[hP_DG_z^{|W|}(p_z[g]) \prod_{z'\in W} p_z(z')\Big] \end{aligned} \tag{20}$$

where

$$\delta_{Z=\phi} \triangleq \begin{cases} 1, & \text{if } Z = \phi \\ 0, & \text{otherwise} \end{cases}. \tag{21}$$

- The notation $\mathcal{P}\angle Z$ denotes that $\mathcal{P}$ partitions the measurement set Z and when used under a summation sign it means that the summation is over all such partitions $\mathcal{P}$.
- The value $|\mathcal{P}|$ denotes the number of sets in the partition $\mathcal{P}$.
- The sets in a partition $\mathcal{P}$ are denoted by the $W \in \mathcal{P}$. When used under a summation sign, it means that the summation is over all the sets in the partition $\mathcal{P}$.
- The value $|W|$ denotes the number of measurements (i.e., the cardinality) in the set W.

Proof: Proof is given in Appendix A for the sake of clarity.□

We can now write the derivative of $F[g,h]$ as

$$\begin{aligned} \frac{\delta}{\delta Z}F[g,h] =& \sum_{S\subseteq Z} \frac{\delta}{\delta(Z-S)} G_{FA}(p_{FA}[g]) \\ & \times \frac{\delta}{\delta S}G_{k|k-1}\Big(p_{k|k-1}\big[h(1 - P_D + P_DG_z(p_z[g]))\big]\Big). \end{aligned} \tag{22}$$

Substituting the result of Theorem 3.1 into (22), we get the following result.

Theorem 3.2 (Derivative of $F[g,h]$ with respect to Z): The derivative of $F[g,h]$ is given as

$$\begin{aligned} \frac{\delta}{\delta Z}F[g,h] =& \left(\prod_{z'\in Z} p_{FA}(z')\right) \sum_{\mathcal{P}\angle Z}\sum_{W\in\mathcal{P}} \Bigg(G_{FA}(p_{FA}[g]) \\ & \times G_{k|k-1}^{(|\mathcal{P}|)}\Big(p_{k|k-1}\big[h(1 - P_D + P_DG_z(p_z[g]))\big]\Big) \\ & \times \frac{1}{|\mathcal{P}|}\eta_W[g,h] + G_{FA}^{(|W|)}(p_{FA}[g]) \\ & \times G_{k|k-1}^{(|\mathcal{P}|-1)}\Big(p_{k|k-1}\big[h(1 - P_D + P_DG_z(p_z[g]))\big]\Big)\Bigg) \\ & \times \prod_{W'\in\mathcal{P}-W} \eta_{W'}[g,h] \end{aligned} \tag{23}$$

where

$$\eta_W[g,h] \triangleq p_{k|k-1}\Big[hP_DG_z^{(|W|)}(p_z[g]) \prod_{z'\in W} \frac{p_z(z')}{p_{FA}(z')}\Big]. \tag{24}$$

□

Proof: Proof is given in Appendix B for the sake of clarity.□

Substituting $g = 0$ in the result of Theorem 3.2, we get

$$\begin{aligned} \frac{\delta}{\delta Z}F[0,h] =& \left(\prod_{z'\in Z} p_{FA}(z')\right) \sum_{\mathcal{P}\angle Z}\sum_{W\in\mathcal{P}} \Bigg(G_{FA}(0) \\ & \times G_{k|k-1}^{(|\mathcal{P}|)}\Big(p_{k|k-1}\big[h(1 - P_D + P_DG_z(0))\big]\Big) \\ & \times \frac{1}{|\mathcal{P}|}\eta_W[0,h] + G_{FA}^{(|W|)}(0) \\ & \times G_{k|k-1}^{(|\mathcal{P}|-1)}\Big(p_{k|k-1}\big[h(1 - P_D + P_DG_z(0))\big]\Big)\Bigg) \\ & \times \prod_{W'\in\mathcal{P}-W} \eta_{W'}[0,h]. \end{aligned} \tag{25}$$

In order to simplify the expressions, we here define the quantity $\rho[h]$ as

$$\rho[h] \triangleq p_{k|k-1}\big[h(1 - P_D + P_DG_z(0))\big] \tag{26}$$

which leads to

$$\frac{\delta}{\delta Z}F[0,h] = \left(\prod_{z'\in Z} p_{FA}(z')\right)\sum_{\mathcal{P}\angle Z}\sum_{W\in\mathcal{P}}\left(G_{FA}(0)\right.$$
$$\left.\times G^{(|\mathcal{P}|)}_{k|k-1}\big(\rho[h]\big)\frac{\eta_W[0,h]}{|\mathcal{P}|} + G^{(|W|)}_{FA}(0)G^{(|\mathcal{P}|-1)}_{k|k-1}\big(\rho[h]\big)\right)$$
$$\times \prod_{W'\in\mathcal{P}-W}\eta_{W'}[0,h]. \tag{27}$$

Taking the derivative with respect to x, we obtain the following result.

Theorem 3.3 (Derivative of $\frac{\delta F[0,h]}{\delta Z}$ with respect to x): The derivative $\frac{\delta}{\delta x}\frac{\delta F[0,h]}{\delta Z}$ is given as

$$\frac{\delta}{\delta x}\frac{\delta F[0,h]}{\delta Z} = \left(\prod_{z'\in Z} p_{FA}(z')\right)p_{k|k-1}(x)$$
$$\times\sum_{\mathcal{P}\angle Z}\sum_{W\in\mathcal{P}}\left(\prod_{W'\in\mathcal{P}-W}\eta_{W'}[0,h]\right)$$
$$\times\left(\left(G_{FA}(0)G^{(|\mathcal{P}|+1)}_{k|k-1}(\rho[h])\frac{\eta_W[0,h]}{|\mathcal{P}|}\right.\right.$$
$$\left.+G^{(|W|)}_{FA}(0)G^{(|\mathcal{P}|)}_{k|k-1}(\rho[h])\right)\big(1-P_D(x)+P_D(x)G_z(0)\big)$$
$$+G_{FA}(0)G^{(|\mathcal{P}|)}_{k|k-1}(\rho[h])\frac{h(x)P_D(x)G^{|W|}_z(0)}{|\mathcal{P}|}\prod_{z'\in W}\frac{p_z(z'|x)}{p_{FA}(z')}$$
$$+\left(G_{FA}(0)G^{(|\mathcal{P}|)}_{k|k-1}(\rho[h])\frac{\eta_W[0,h]}{|\mathcal{P}|}\right.$$
$$\left.+G^{(|W|)}_{FA}(0)G^{(|\mathcal{P}|-1)}_{k|k-1}(\rho[h])\right)$$
$$\left.\times\sum_{W'\in\mathcal{P}-W}\frac{h(x)P_D(x)G^{|W'|}_z(0)}{\eta_{W'}[0,h]}\prod_{z'\in W'}\frac{p_z(z'|x)}{p_{FA}(z')}\right). \tag{28}$$

□

Proof: Proof is given in Appendix C for the sake of clarity.□

If we substitute $h = 1$ into the result of Theorem 3.3, we get

$$\frac{\delta}{\delta x}\frac{\delta}{\delta Z}F[0,1] = \left(\prod_{z'\in Z} p_{FA}(z')\right)p_{k|k-1}(x)$$
$$\times\sum_{\mathcal{P}\angle Z}\sum_{W\in\mathcal{P}}\left(\prod_{W'\in\mathcal{P}-W}\eta_{W'}[0,1]\right)$$
$$\times\left(\left(G_{FA}(0)G^{(|\mathcal{P}|+1)}_{k|k-1}(\rho[1])\frac{\eta_W[0,1]}{|\mathcal{P}|}\right.\right.$$
$$\left.+G^{(|W|)}_{FA}(0)G^{(|\mathcal{P}|)}_{k|k-1}(\rho[1])\right)\big(1-P_D(x)+P_D(x)G_z(0)\big)$$
$$+G_{FA}(0)G^{(|\mathcal{P}|)}_{k|k-1}(\rho[1])\frac{P_D(x)G^{|W|}_z(0)}{|\mathcal{P}|}\prod_{z'\in W}\frac{p_z(z'|x)}{p_{FA}(z')}$$
$$+\left(G_{FA}(0)G^{(|\mathcal{P}|)}_{k|k-1}(\rho[1])\frac{\eta_W[0,1]}{|\mathcal{P}|}\right.$$
$$\left.+G^{(|W|)}_{FA}(0)G^{(|\mathcal{P}|-1)}_{k|k-1}(\rho[1])\right)$$
$$\left.\times\sum_{W'\in\mathcal{P}-W}\frac{P_D(x)G^{|W'|}_z(0)}{\eta_{W'}[0,1]}\prod_{z'\in W'}\frac{p_z(z'|x)}{p_{FA}(z')}\right). \tag{29}$$

In order to simplify the result, we define the constants

$$\beta_{\mathcal{P},W} \triangleq G_{FA}(0)G^{(|\mathcal{P}|)}_{k|k-1}(\rho[1])\frac{\eta_W[0,1]}{|\mathcal{P}|} + G^{(|W|)}_{FA}(0)G^{(|\mathcal{P}|-1)}_{k|k-1}(\rho[1]), \tag{30}$$

$$\gamma_{\mathcal{P},W} \triangleq G_{FA}(0)G^{(|\mathcal{P}|+1)}_{k|k-1}(\rho[1])\frac{\eta_W[0,1]}{|\mathcal{P}|} + G^{(|W|)}_{FA}(0)G^{(|\mathcal{P}|)}_{k|k-1}(\rho[1]), \tag{31}$$

$$\alpha_{\mathcal{P},W} \triangleq \prod_{W'\in\mathcal{P}-W}\eta_{W'}[0,1]. \tag{32}$$

We can now write the above results in terms of these constants as follows.

$$\frac{\delta}{\delta x}\frac{\delta}{\delta Z}F[0,1] = \left(\prod_{z'\in Z} p_{FA}(z')\right)p_{k|k-1}(x)$$
$$\times\sum_{\mathcal{P}\angle Z}\sum_{W\in\mathcal{P}}\alpha_{\mathcal{P},W}\left(\gamma_{\mathcal{P},W}\big(1-P_D(x)+P_D(x)G_z(0)\big)\right.$$
$$+G_{FA}(0)G^{(|\mathcal{P}|)}_{k|k-1}(\rho[1])\frac{P_D(x)G^{|W|}_z(0)}{|\mathcal{P}|}\prod_{z'\in W}\frac{p_z(z'|x)}{p_{FA}(z')}$$
$$\left.+\beta_{\mathcal{P},W}\sum_{W'\in\mathcal{P}-W}\frac{P_D(x)G^{|W'|}_z(0)}{\eta_{W'}[0,1]}\prod_{z'\in W'}\frac{p_z(z'|x)}{p_{FA}(z')}\right) \tag{33}$$

$$\frac{\delta}{\delta Z}F[0,1] = \left(\prod_{z'\in Z} p_{FA}(z')\right)\sum_{\mathcal{P}\angle Z}\sum_{W\in\mathcal{P}}\alpha_{\mathcal{P},W}\beta_{\mathcal{P},W}. \tag{34}$$

Dividing the two quantities above, we obtain the updated PHD $D_{k|k}(\cdot)$ as given below.

$$D_{k|k}(x) = \frac{\sum_{\mathcal{P}\angle Z}\sum_{W\in\mathcal{P}}\alpha_{\mathcal{P},W}\gamma_{\mathcal{P},W}}{\sum_{\mathcal{P}\angle Z}\sum_{W\in\mathcal{P}}\alpha_{\mathcal{P},W}\beta_{\mathcal{P},W}}$$
$$\times(1-P_D(x)+P_D(x)G_z(0))p_{k|k-1}(x)$$
$$+\frac{\left(\begin{array}{c}\sum_{\mathcal{P}\angle Z}\sum_{W\in\mathcal{P}}\alpha_{\mathcal{P},W}\Big(G_{FA}(0)G^{(|\mathcal{P}|)}_{k|k-1}(\rho[1])\\ \times\frac{P_D(x)G^{|W|}_z(0)}{|\mathcal{P}|}\prod_{z'\in W}\frac{p_z(z'|x)}{p_{FA}(z')}\\ +\beta_{\mathcal{P},W}\sum_{W'\in\mathcal{P}-W}\frac{P_D(x)G^{|W'|}_z(0)}{\eta_{W'}[0,1]}\prod_{z'\in W'}\frac{p_z(z'|x)}{p_{FA}(z')}\Big)\end{array}\right)}{\sum_{\mathcal{P}\angle Z}\sum_{W\in\mathcal{P}}\alpha_{\mathcal{P},W}\beta_{\mathcal{P},W}}$$
$$\times p_{k|k-1}(x) \tag{35}$$
$$=\frac{\sum_{\mathcal{P}\angle Z}\sum_{W\in\mathcal{P}}\alpha_{\mathcal{P},W}\gamma_{\mathcal{P},W}}{\sum_{\mathcal{P}\angle Z}\sum_{W\in\mathcal{P}}\alpha_{\mathcal{P},W}\beta_{\mathcal{P},W}}$$
$$\times(1-P_D(x)+P_D(x)G_z(0))p_{k|k-1}(x)$$

$$+\frac{\begin{pmatrix}\sum_{\mathcal{P}\angle Z}\sum_{W\in\mathcal{P}} G_z^{|W|}(0)\\ \times\Big(\frac{\alpha_{\mathcal{P},W}}{|\mathcal{P}|}G_{FA}(0)G_{k|k-1}^{(|\mathcal{P}|)}(\rho[1])\\ +\frac{\sum_{W'\in\mathcal{P}-W}\alpha_{\mathcal{P},W'}\beta_{\mathcal{P},W'}}{\eta_W[0,1]}\Big)\end{pmatrix}}{\sum_{\mathcal{P}\angle Z}\sum_{W\in\mathcal{P}}\alpha_{\mathcal{P},W}\beta_{\mathcal{P},W}}$$
$$\times P_D(x)\prod_{z'\in W}\frac{p_z(z'|x)}{p_{FA}(z')}p_{k|k-1}(x). \quad (36)$$

If we define the additional coefficients κ and $\sigma_{\mathcal{P},W}$ as

$$\kappa \triangleq \frac{\sum_{\mathcal{P}\angle Z}\sum_{W\in\mathcal{P}}\alpha_{\mathcal{P},W}\gamma_{\mathcal{P},W}}{\sum_{\mathcal{P}\angle Z}\sum_{W\in\mathcal{P}}\alpha_{\mathcal{P},W}\beta_{\mathcal{P},W}}, \quad (37)$$

$$\sigma_{\mathcal{P},W} \triangleq G_z^{|W|}(0)\bigg(\frac{\alpha_{\mathcal{P},W}}{|\mathcal{P}|}G_{FA}(0)G_{k|k-1}^{(|\mathcal{P}|)}(\rho[1]) + \frac{\sum_{W'\in\mathcal{P}-W}\alpha_{\mathcal{P},W'}\beta_{\mathcal{P},W'}}{\eta_W[0,1]}\bigg), \quad (38)$$

we can obtain the final PHD update equation as in (39) on the next page. We give a proof that (39) reduces to the ETT-PHD update formula of [6] in the case of Poisson prior, false alarms and target generated measurements in Appendix D.

Substituting $h(x) = x$ in the result of Theorem 3.2, we get

$$\frac{\delta}{\delta Z}F[0,x] = \bigg(\prod_{z'\in Z}p_{FA}(z')\bigg)\sum_{\mathcal{P}\angle Z}\sum_{W\in\mathcal{P}}\alpha_{\mathcal{P},W}\bigg(G_{FA}(0)\times G_{k|k-1}^{(|\mathcal{P}|)}(\rho[1]x)\frac{\eta_W[0,1]}{|\mathcal{P}|}x^{|\mathcal{P}|} + G_{FA}^{(|W|)}(0)G_{k|k-1}^{(|\mathcal{P}|-1)}(\rho[1]x)x^{|\mathcal{P}|-1}\bigg). \quad (40)$$

Now dividing by $\frac{\delta}{\delta Z}F[0,1]$ which is given in (34), we get the posterior p.g.f. $G_{k|k}(\cdot)$ of the target number given in (41) on the next page. Taking the nth derivatives with respect to x, evaluating at $x = 0$ and dividing by $n!$, would give posterior probability mass function $P_{k|k}(n)$ of the target number given in (42) on the next page.

IV. A Gaussian Mixture Implementation

In this section, we assume the following.

- The prior PHD, $D_{k|k-1}(\cdot)$ is a Gaussian mixture given as

$$D_{k|k-1}(x) = \sum_{j=1}^{J_{k|k-1}} w_{k|k-1}^j\mathcal{N}(x; m_{k|k-1}^j, P_{k|k-1}^j); \quad (43)$$

- The individual measurements belonging to targets are related to the corresponding target state according to the measurement equation

$$z_k = Cx_k + v_k \quad (44)$$

  where $v_k \sim \mathcal{N}(0,R)$ is the measurement noise. In this case, we have the Gaussian individual measurement likelihood $p_z(\cdot|x) = \mathcal{N}(\cdot; Cx, R)$.

- The following approximation about the detection probability function $P_D(x)$ holds.

$$P_D(x)\mathcal{N}(x;m,P) \approx P_D(m)\mathcal{N}(x;m,P) \quad (45)$$

  where $m \in \mathbb{R}^{n_x}$ and $P \in \mathbb{R}^{n_x\times n_x}$ are arbitrary mean and covariances. This approximation is made for the sake of simplicity and for not being simplistic by assuming a constant probability of detection. The formulations can be generalized to a function $P_D(\cdot)$ represented by a Gaussian mixture straightforwardly though this would increase the complexity significantly.

The Gaussian mixture implementation we propose has the following steps.

1) Calculate $\rho[1]$ as

$$\rho[1] = \sum \bar{w}_{k|k-1}^j(1 - P_D^j + P_D^jG_z(0)) \quad (46)$$

   where $P_D^j \triangleq P_D(m_{k|k-1}^j)$ and

$$\bar{w}_{k|k-1}^j \triangleq \frac{w_{k|k-1}^j}{\sum_{\ell=1}^{J_{k|k-1}} w_{k|k-1}^\ell} \quad (47)$$

   for $j = 1,\ldots,J_{k|k-1}$ are the normalized prior PHD coefficients.

2) Calculate $\eta_W[0,1]$ for all sets W in all partitions $\mathcal{P}$ of Z_k as follows.

$$\eta_W[0,1] = G_z^{(|W|)}(0)\sum_{j=1}^{J_{k|k-1}}\bar{w}_{k|k-1}^jP_D^j\frac{L_z^{j,W}}{L_{FA}^W} \quad (48)$$

   where

$$L_z^{j,W} = \mathcal{N}(\mathbf{z}_W; \mathbf{z}_{k|k-1}^{j,W}, \mathbf{S}_{k|k-1}^{j,W}) \quad (49)$$
$$L_{FA}^W = \prod_{z\in W}p_{FA}(z) \quad (50)$$
$$\mathbf{z}_{k|k-1}^{j,W} = \mathbf{C}_Wm_{k|k-1}^j \quad (51)$$
$$\mathbf{S}_{k|k-1}^{j,W} = \mathbf{C}_WP_{k|k-1}^j\mathbf{C}_W^{\mathrm{T}} + \mathbf{R}_W \quad (52)$$
$$\mathbf{z}_W \triangleq \bigoplus_{z\in W} z, \quad (53)$$
$$\mathbf{C}_W = [\underbrace{C^{\mathrm{T}}, C^{\mathrm{T}}, \cdots, C^{\mathrm{T}}}_{|W|\text{ times}}]^{\mathrm{T}}, \quad (54)$$
$$\mathbf{R}_W = \mathrm{blkdiag}(\underbrace{R, R, \cdots, R}_{|W|\text{ times}}). \quad (55)$$

   The operation $\bigoplus$ denotes vertical vectorial concatenation.

3) Calculate the coefficients $\beta_{\mathcal{P},W}$, $\gamma_{\mathcal{P},W}$, $\alpha_{\mathcal{P},W}$, κ and $\sigma_{\mathcal{P},W}$ for all sets W and all partitions $\mathcal{P}$ using the formulas (30), (31), (32), (37) and (38) respectively.

4) Calculate the posterior means $m_{k|k}^{j,W}$ and covariances $P_{k|k}^{j,W}$ as

$$m_{k|k}^{j,W} = m_{k|k-1}^j + \mathbf{K}^{j,W}(\mathbf{z}_W - \mathbf{z}_{k|k-1}^{j,W}) \quad (56)$$
$$P_{k|k}^{j,W} = P_{k|k-1}^j - \mathbf{K}^{j,W}\mathbf{S}_{k|k-1}^{j,W}(\mathbf{K}^{j,W})^{\mathrm{T}} \quad (57)$$
$$\mathbf{K}^{j,W} \triangleq P_{k|k-1}^j\mathbf{C}_W^{\mathrm{T}}\left(\mathbf{S}_{k|k-1}^{j,W}\right)^{-1}. \quad (58)$$

$$D_{k|k}(x) = \left(\kappa(1 - P_D(x) + P_D(x)G_z(0)) + \frac{\sum_{\mathcal{P}\angle Z}\sum_{W\in\mathcal{P}} \sigma_{\mathcal{P},W} \prod_{z'\in W} \frac{p_z(z'|x)}{p_{FA}(z')}}{\sum_{\mathcal{P}\angle Z}\sum_{W\in\mathcal{P}} \alpha_{\mathcal{P},W}\beta_{\mathcal{P},W}} P_D(x) \right) p_{k|k-1}(x). \tag{39}$$

$$G_{k|k}(x) = \frac{\sum_{\mathcal{P}\angle Z}\sum_{W\in\mathcal{P}} \alpha_{\mathcal{P},W} \left(G_{FA}(0) G_{k|k-1}^{(|\mathcal{P}|)}(\rho[1]x) \frac{\eta_W[0,1]}{|\mathcal{P}|} x^{|\mathcal{P}|} + G_{FA}^{(|W|)}(0) G_{k|k-1}^{(|\mathcal{P}|-1)}(\rho[1]x) x^{|\mathcal{P}|-1} \right)}{\sum_{\mathcal{P}\angle Z}\sum_{W\in\mathcal{P}} \alpha_{\mathcal{P},W}\beta_{\mathcal{P},W}}; \tag{41}$$

$$P_{k|k}(n) = \frac{\sum_{\mathcal{P}\angle Z}\sum_{W\in\mathcal{P}} \alpha_{\mathcal{P},W} G_{k|k-1}^{(n)}(0) \left(G_{FA}(0) \frac{\eta_W[0,1]}{|\mathcal{P}|} \frac{\rho[1]^{n-|\mathcal{P}|}}{(n-|\mathcal{P}|)!} \delta_{n\geq|\mathcal{P}|} + G_{FA}^{(|W|)}(0) \frac{\rho[1]^{n-|\mathcal{P}|+1}}{(n-|\mathcal{P}|+1)!} \delta_{n\geq|\mathcal{P}|-1} \right)}{\sum_{\mathcal{P}\angle Z}\sum_{W\in\mathcal{P}} \alpha_{\mathcal{P},W}\beta_{\mathcal{P},W}}. \tag{42}$$

5) Calculate the posterior weights $w_{k|k}^{j,\mathcal{P},W}$ as

$$w_{k|k}^{j,\mathcal{P},W} = \frac{\bar{w}_{k|k-1}^{j} P_D^j \sigma_{\mathcal{P},W} \frac{L_z^{j,W}}{L_{FA}^W}}{\sum_{\mathcal{P}\angle Z}\sum_{W\in\mathcal{P}} \alpha_{\mathcal{P},W}\beta_{\mathcal{P},W}}. \tag{59}$$

After these steps, the posterior PHD can be calculated as

$$\begin{aligned} D_{k|k}(x) =& \kappa \sum_{j=1}^{J_{k|k-1}} \bar{w}_{k|k-1}^{j} (1 - P_D^j + P_D^j G_z(0)) \\ & \times \mathcal{N}(x; m_{k|k-1}^j, P_{k|k-1}^j) \\ & + \sum_{\mathcal{P}\angle Z_k} \sum_{W\in\mathcal{P}} w_{k|k}^{j,\mathcal{P},W} \mathcal{N}(x; m_{k|k}^{j,W}, P_{k|k}^{j,W}) \end{aligned} \tag{60}$$

which has

$$J_{k|k} = J_{k|k-1} \left(1 + \sum_{\mathcal{P}\angle Z_k} |\mathcal{P}| \right) \tag{61}$$

components in total. The number of components can further be decreased by identifying the identical sets W in different partitions and combining the corresponding components by including only one Gaussian with weight equal to the sum of the weights of the previous components. The usual techniques of merging and pruning should still be applied to reduce the exponential growing of the number of components, see [5] for details.

The calculation of the updated cardinality distribution $P_{k|k}(\cdot)$ is straightforward with (42) using the quantities calculated above.

Note that the ETT-CPHD filter, like ETT-PHD, requires all partitions of the current measurement set for its update. This makes these filters computationally infeasible even with the toy examples and hence approximations are necessary. We are going to use the partitioning algorithm presented in [7] to solve this problem efficiently. This partitioning algorithm basically puts constraints on the distances between the measurements in a set of a partition and reduces the number of partitions to be considered significantly sacrificing as little performance as possible. The details of the partitioning algorithm are not given here due to space considerations and the reader is referred to [7].

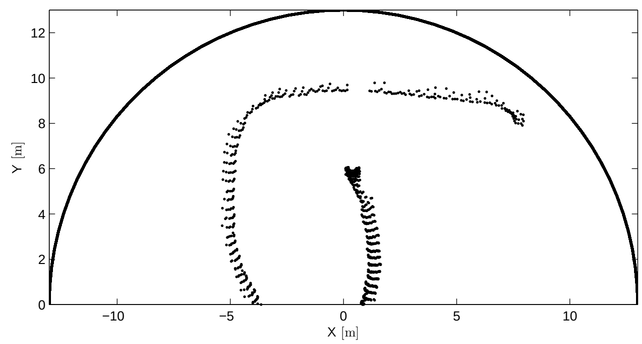

Figure 1. Surveillance region and laser sensor measurements.

V. Experimental Results

In this section, we illustrate the early results of an experiment made with the ETT-CPHD filter using a laser sensor and compare it to the ETT-PHD version in terms of estimated number of targets. In the experiment the measurements were collected using a SICK LMS laser range sensor. The sensor measures range every $0.5°$ over a $180°$ surveillance area. Ranges shorter than $13\ \mathrm{m}$ were converted to (x, y) measurements using a polar to Cartesian transformation. The data set contains 100 laser range sweeps in total. During the data collection two humans moved through the surveillance area, entering the surveillance area at different times. The laser sensor was at the waist level of the humans and each human was giving rise to, on average, 10 clustered laser returns. The first human enters the surveillance area at time $k = 22$, and moves to the center of the surveillance area where he remains still until the end of the experiment. The second human enters at time $k = 38$, and proceeds to move behind the first target, thus both entering and exiting an occluded part of the surveillance area. We illustrate the surveillance region and the collected measurements in Figure 1.

Since there is no ground truth available it is difficult to obtain a definite measure of target tracking quality, however by examining the raw data we were able to observe the true cardinality (0 from time $k = 1$ to $k = 21$, 1 from time $k = 22$

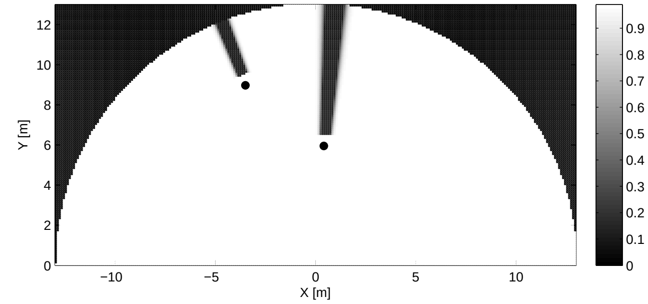

Figure 2. The function $P_D(\cdot)$ at time $k = 65$.

to $k = 37$ and 2 from time $k = 38$ to $k = 100$), which can thus be compared to the estimated cardinality.

When the probability of detection $P_D(\cdot)$ is kept constant over all of the surveillance area, the target loss is evident in this scenario when the second human is occluded by the first since the tracker always expects to detect the targets. Avoiding such a loss is possible using inhomogeneous detection probability over the surveillance region based on the target estimates. The knowledge of the targets that are present, i.e., the estimated Gaussian components of the PHD, can be used to determine which parts of the surveillance area are likely to be occluded and which parts are not. The estimated range r and bearing φ between the sensor and the target may be computed from the state variables. The P_D^j values of the components behind each component, i.e., the components at a larger range from the sensor than a target, is reduced from a nominal value P_D^0 according to the weight and bearing standard deviation of the Gaussian component. The exact reduction expression is quite complicated and omitted here. Instead we give a pictorial illustration of the function $P_D(\cdot)$ that we use in Fig 2.

We run both ETT-PHD and ETT-CPHD filters with (nearly) constant velocity target dynamic models that have zero mean white Gaussian distributed acceleration noise of $2\ \mathrm{m/s^2}$ standard deviation. The measurement noise is assumed to be Gaussian distributed with zero-mean and $0.1\ \mathrm{m}$ standard deviation. Uniform Poisson false alarms are assumed. Since there is almost no false alarm in the scenario, the false alarm rate has been set to be $1/V_S$, where V_S is the area of the surveillance region, which corresponds to only a single false alarm per scan. Both algorithms use Poisson target generated measurements with uniform rate of 12 measurements per scan. Notice that, with these settings, the ETT-CPHD filter is different than ETT-PHD filter only in that its posterior (and prior) is a cluster process rather than a Poisson process which is the case for ETT-PHD filter.

The first tracking experiment was performed with the variable probability of detection described above with the nominal detection probability of $P_D^0 = 0.99$. The ETT-CPHD filter obtained its cardinality estimates as the maximum-a-posteriori (MAP) estimates of the cardinality probability mass function. The ETT-CPHD filter set the cardinality estimates based on the rounded values of the Gaussian mixture PHD

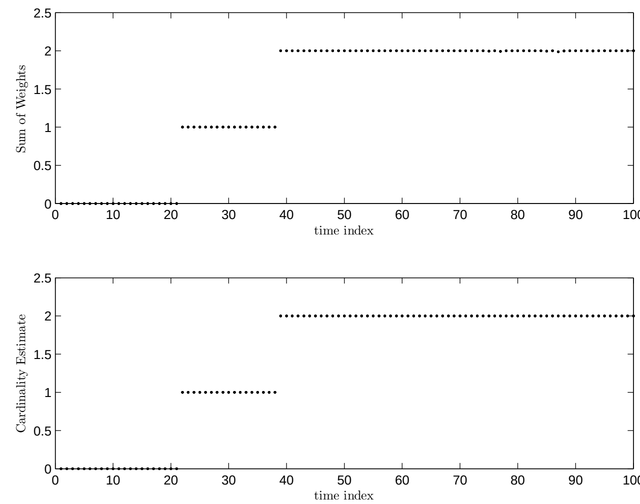

Figure 3. The sum of weights (upper figure) and the cardinality estimates (lower figure) of the ETT-CPHD filter when nominal probability of detection is $P_D^0 = 0.99$.

weights. The sum of the Gaussian mixture PHD weights of ETT-CPHD and ETT-PHD algorithms along with their corresponding cardinality estimates are shown in Figures 3 and 4 respectively. Although both of the algorithms cardinality estimates are the same, their sum of PHD weights differ especially during the occlusion of one of the targets. ETT-PHD filter has been discovered to have problems especially when the target occlusion ends. When the target appears into the view of the sensor from behind the other target, its detection probability still remains slightly lower than the nominal probability of detection. The ETT-PHD filters sum of weights tends to grow as soon as several measurements of the occluded target appear. This strange phenomenon can be explained with the following basic example: Assuming no false alarms and a single target with existence probability P_{E}, a single detection (without any other information than the detection itself) should cause the expected number of targets to be exactly unity. However, applying the standard PHD formulae, one can calculate this number to be $1 + P_{\mathrm{E}}(1 - P_D^j)$ whose bias increases as P_D^j decreases. We have seen that when the target exits the occluded region, the sudden increase in the sum of weights appearing is a manifestation of this type of sensitivity of the PHD filter. A similar sensitivity issue is mentioned in [12] for the case of no detection. The ETT-CPHD filter on the other hand shows a perfect performance in this case.

In order to further examine the stability issues depending on low values of the detection probability, in the second tracking experiment, we set the nominal probability of detection to a slightly lower value $P_D^0 = 0.7$. The results for the ETT-CPHD and ETT-PHD filters are illustrated in Figures 5 and 6. respectively. We see that while the ETT-CPHD filters performance hardly changes, the ETT-PHD filter performance shows remarkable differences. When the occlusion ends, the

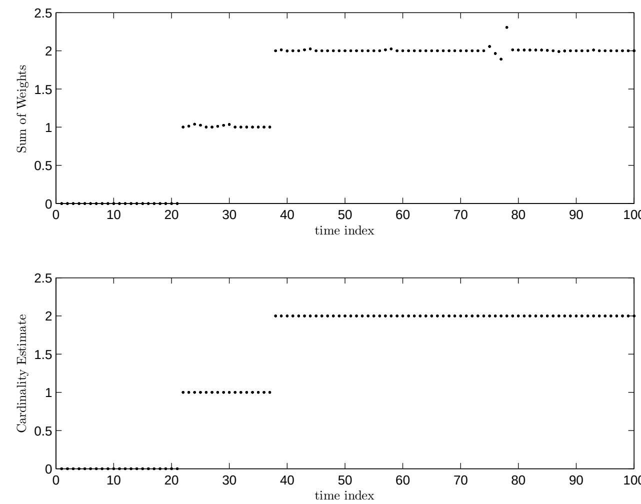

Figure 4. The sum of weights (upper figure) and the cardinality estimates (lower figure) of the ETT-PHD filter when nominal probability of detection is $P_D^0 = 0.99$.

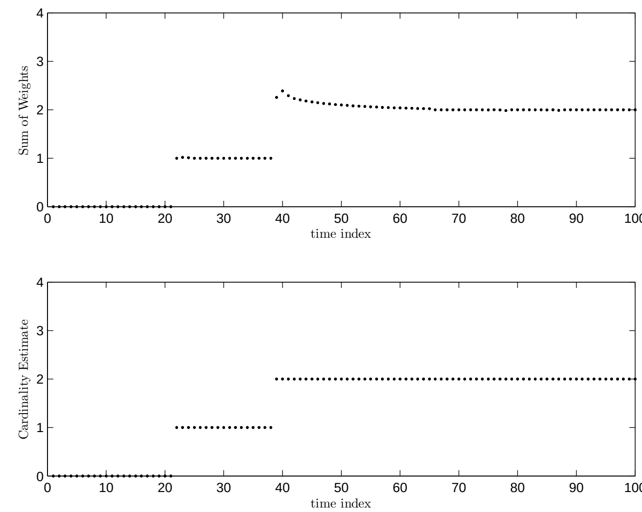

Figure 5. The sum of weights (upper figure) and the cardinality estimates (lower figure) of the ETT-CPHD filter when nominal probability of detection is $P_D^0 = 0.7$.

jump in the sum of weights is so large for ETT-PHD that the cardinality estimates are also affected. Another important change is that ETT-PHD has significantly biased sum of PHD weights in the steady state. This phenomenon has the following explanation in the case of the basic example discussed above. When consecutive detections are obtained when P_D^j is low, the PHD weight would converge to the fixed point of the equation

$$P_{\mathrm{E}} = 1 + P_{\mathrm{E}}(1 - P_D^j) \tag{62}$$

which is $P_E = \frac{1}{P_D^j}$. In this case, since $P_D^0 = 0.7$, the sum of weights is converging approximately to $1/0.7 \approx 1.43$ and $2/0.7 \approx 2.85$ for the single and two target cases respectively. The sum of ETT-CPHD filters PHD weights on the other hand converge to much more reasonable values compared to those of ETT-PHD.

When we made further trials, it has been seen that ETT-CPHD filter is not immune to such issues either when the nominal probability of detection P_D^0 values are decreased further. However, it seems to be much more robust to this phenomenon than ETT-PHD the filter.

VI. Conclusions

A CPHD filter has been derived for the extended targets which can give rise to multiple measurements per each scan modeled by an i.i.d. cluster process. A Gaussian mixture implementation for the derived filter has also been proposed. The results of early experiments on laser data show that the cardinality estimates and the PHD weights of the new ETT-CPHD filter is more robust to different parameter settings than its PHD counterpart.

The experiments made on the laser data contained few (if not none) false alarms. Further experiments with significant clutter must be made to evaluate performance improvements compared to the ETT-PHD filter.

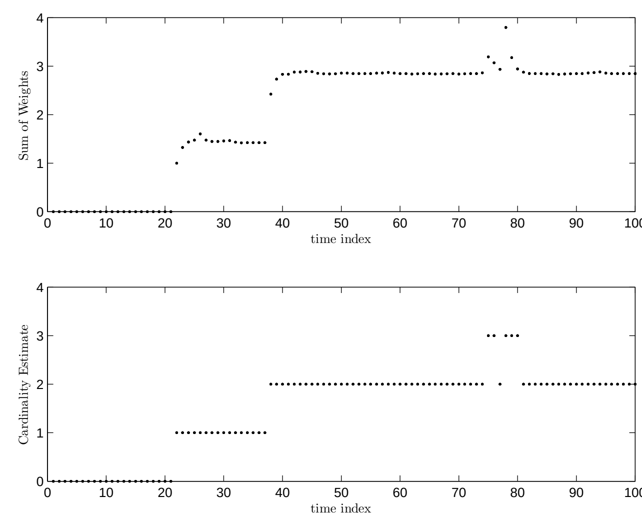

Figure 6. The sum of weights (upper figure) and the cardinality estimates (lower figure) of the ETT-PHD filter when nominal probability of detection is $P_D^0 = 0.7$.

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Appendix A
Proof of Theorem 3.1

Proof is done by induction.

- Let $Z=\phi$, then

$$\frac{\delta}{\delta Z}G_{k|k-1}\Big(p_{k|k-1}\big[h(1-P_D+P_DG_z(p_z[g]))\big]\Big) = G_{k|k-1}\Big(p_{k|k-1}\big[h(1-P_D+P_DG_z(p_z[g]))\big]\Big) \quad (63)$$

by the definition of functional (set) derivative.

- Let $Z_1=\{z_1\}$, then

$$\frac{\delta}{\delta z_1}G_{k|k-1}\Big(p_{k|k-1}\big[h(1-P_D+P_DG_z(p_z[g]))\big]\Big) = G^{(1)}_{k|k-1}\Big(p_{k|k-1}\big[h(1-P_D+P_DG_z(p_z[g]))\big]\Big) \times p_{k|k-1}\big[p_z[g])hG^{(1)}_z P_D p_z(z_1)\big] \quad (64)$$

which together with the case $Z=\phi$ proves that

$$\frac{\delta}{\delta Z}G_{k|k-1}\Big(p_{k|k-1}\big[h(1-P_D+P_DG_z(p_z[g]))\big]\Big) = G_{k|k-1}\Big(p_{k|k-1}\big[h(1-P_D+P_DG_z(p_z[g]))\big]\Big)\delta_{Z=\phi} + G^{(1)}_{k|k-1}\Big(p_{k|k-1}\big[h(1-P_D+P_DG_z(p_z[g]))\big]\Big) \times p_{k|k-1}\big[p_z[g])hG^{(1)}_z P_D p_z(z_1)\big] \quad (65)$$

for $Z=\phi$ and $Z=\{z_1\}$.

Now assume that the result of Theorem 3.1 holds for $Z_{n-1}=\{z_1,\ldots,z_{n-1}\}$, $n\geq 2$. Then we are going to see whether the result holds of $Z_n=Z_{n-1}\cup z_n$.

$$\frac{\delta}{\delta Z_{n-1}\cup z_n}G_{k|k-1}\Big(p_{k|k-1}\big[h(1-P_D+P_DG_z(p_z[g]))\big]\Big) = \frac{\delta}{\delta z_n}\frac{\delta}{\delta Z_{n-1}}G_{k|k-1}\Big(p_{k|k-1}\big[h(1-P_D+P_DG_z(p_z[g]))\big]\Big) \quad (66)$$

$$= \frac{\delta}{\delta z_n}\Bigg(G_{k|k-1}\Big(p_{k|k-1}\big[h(1-P_D+P_DG_z(p_z[g]))\big]\Big)\delta_{Z_{n-1}=\phi} + \sum_{\mathcal{P}\angle Z_{n-1}} G^{(|\mathcal{P}|)}_{k|k-1}\Big(p_{k|k-1}\big[h(1-P_D+P_DG_z(p_z[g]))\big]\Big) \times \prod_{W\in\mathcal{P}} p_{k|k-1}\Big[hP_DG^{|W|}_z(p_z[g])\prod_{z'\in W}p_z(z')\Big]\Bigg) \quad (67)$$

$$= \frac{\delta}{\delta z_n}\sum_{\mathcal{P}\angle Z_{n-1}} G^{(|\mathcal{P}|)}_{k|k-1}\Big(p_{k|k-1}\big[h(1-P_D+P_DG_z(p_z[g]))\big]\Big) \times \prod_{W\in\mathcal{P}} p_{k|k-1}\Big[hP_DG^{|W|}_z(p_z[g])\prod_{z'\in W}p_z(z')\Big] \quad (68)$$

$$= \sum_{\mathcal{P}\angle Z_{n-1}}\Bigg(G^{(|\mathcal{P}|+1)}_{k|k-1}\Big(p_{k|k-1}\big[h(1-P_D+P_DG_z(p_z[g]))\big]\Big) \times p_{k|k-1}\big[hP_DG^{(1)}_z(p_z[g]))p_z(z_n)\big] \times \prod_{W\in\mathcal{P}} p_{k|k-1}\Big[hP_DG^{|W|}_z(p_z[g])\prod_{z'\in W}p_z(z')\Big] + G^{(|\mathcal{P}|)}_{k|k-1}\Big(p_{k|k-1}\big[h(1-P_D+P_DG_z(p_z[g]))\big]\Big) \times \sum_{W\in\mathcal{P}} p_{k|k-1}\Big[hP_DG^{|W|+1}_z(p_z[g])\prod_{z'\in W\cup z_n}p_z(z')\Big] \times \prod_{W'\in\mathcal{P}-W} p_{k|k-1}\Big[hP_DG^{|W'|}_z(p_z[g])\prod_{z'\in W'}p_z(z')\Big]\Bigg) \quad (69)$$

$$= \sum_{\mathcal{P}\angle Z_{n-1}}\Bigg(G^{(|\mathcal{P}|+1)}_{k|k-1}\Big(p_{k|k-1}\big[h(1-P_D+P_DG_z(p_z[g]))\big]\Big) \times \prod_{W\in\mathcal{P}\cup\{z_n\}} p_{k|k-1}\Big[hP_DG^{|W|}_z(p_z[g])\prod_{z'\in W}p_z(z')\Big] + G^{(|\mathcal{P}|)}_{k|k-1}\Big(p_{k|k-1}\big[h(1-P_D+P_DG_z(p_z[g]))\big]\Big) \times \sum_{W\in\mathcal{P}}\prod_{W'\in\mathcal{P}-W\cup(W\cup z_n)} \times p_{k|k-1}\Big[hP_DG^{|W'|}_z(p_z[g])\prod_{z'\in W'}p_z(z')\Big]\Bigg) \quad (70)$$

$$= \sum_{\mathcal{P}\angle Z_{n-1}\cup z_n} G^{(|\mathcal{P}|)}_{k|k-1}\Big(p_{k|k-1}\big[h(1-P_D+P_DG_z(p_z[g]))\big]\Big) \times \prod_{W\in\mathcal{P}} p_{k|k-1}\Big[hP_DG^{|W|}_z(p_z[g])\prod_{z'\in W}p_z(z')\Big] \quad (71)$$

which is equal to

$$\frac{\delta}{\delta Z_n}G_{k|k-1}\Big(p_{k|k-1}\big[h(1-P_D+P_DG_z(p_z[g]))\big]\Big) \quad (72)$$

by the result of Theorem 3.1. The proof is complete. □

APPENDIX B
PROOF OF THEOREM 3.2

Substituting the result of Theorem 3.1 into (22), we obtain

$$\frac{\delta}{\delta Z}F[g,h] = \sum_{S\subseteq Z}\frac{\delta}{\delta(Z-S)}G_{FA}(p_{FA}[g]) \times \Bigg(G_{k|k-1}\Big(p_{k|k-1}\big[h(1-P_D+P_DG_z(p_z[g]))\big]\Big)\delta_{S=\phi} + \sum_{\mathcal{P}\angle S} G^{(|\mathcal{P}|)}_{k|k-1}\Big(p_{k|k-1}\big[h(1-P_D+P_DG_z(p_z[g]))\big]\Big) \times \prod_{W\in\mathcal{P}} p_{k|k-1}\Big[hP_DG^{|W|}_z(p_z[g])\prod_{z'\in W}p_z(z')\Big]\Bigg) \quad (73)$$

$$= \frac{\delta}{\delta Z}G_{FA}(p_{FA}[g]) \times G_{k|k-1}\Big(p_{k|k-1}\big[h(1-P_D+P_DG_z(p_z[g]))\big]\Big) + \sum_{S\subseteq Z}\frac{\delta}{\delta(Z-S)}G_{FA}(p_{FA}[g]) \times \sum_{\mathcal{P}\angle S} G^{(|\mathcal{P}|)}_{k|k-1}\Big(p_{k|k-1}\big[h(1-P_D+P_DG_z(p_z[g]))\big]\Big) \times \prod_{W\in\mathcal{P}} p_{k|k-1}\Big[hP_DG^{|W|}_z(p_z[g])\prod_{z'\in W}p_z(z')\Big] \quad (74)$$

$$= \Bigg(\prod_{z'\in Z}p_{FA}(z')\Bigg)\Bigg(G^{(|Z|)}_{FA}(p_{FA}[g]) \times G_{k|k-1}\Big(p_{k|k-1}\big[h(1-P_D+P_DG_z(p_z[g]))\big]\Big)$$

$$+\sum_{S\subseteq Z} G_{FA}^{(|Z-S|)}(p_{FA}[g]) \times \sum_{\mathcal{P}\angle S} G_{k|k-1}^{(|\mathcal{P}|)}\Big(p_{k|k-1}\big[h(1-P_D+P_D G_z(p_z[g]))\big]\Big) \times \prod_{W\in\mathcal{P}} p_{k|k-1}\Big[hP_D G_z^{|W|}(p_z[g]) \prod_{z'\in W}\frac{p_z(z')}{p_{FA}(z')}\Big]\Big) \tag{75}$$

$$\begin{aligned}=&\Big(\prod_{z'\in Z} p_{FA}(z')\Big)\Big(G_{FA}^{(|Z|)}(p_{FA}[g]) \qquad (76)\\ &\times G_{k|k-1}\Big(p_{k|k-1}\big[h(1-P_D+P_D G_z(p_z[g]))\big]\Big)\\ &+G_{FA}(p_{FA}[g])\\ &\times\sum_{\mathcal{P}\angle Z} G_{k|k-1}^{(|\mathcal{P}|)}\Big(p_{k|k-1}\big[h(1-P_D+P_D G_z(p_z[g]))\big]\Big)\\ &\times\prod_{W\in\mathcal{P}} p_{k|k-1}\Big[hP_D G_z^{|W|}(p_z[g])\prod_{z'\in W}\frac{p_z(z')}{p_{FA}(z')}\Big]\\ &+\sum_{\substack{S\subseteq Z\\ S\neq Z}} G_{FA}^{(|Z-S|)}(p_{FA}[g])\\ &\times\sum_{\mathcal{P}\angle S} G_{k|k-1}^{(|\mathcal{P}|)}\Big(p_{k|k-1}\big[h(1-P_D+P_D G_z(p_z[g]))\big]\Big)\\ &\times\prod_{W\in\mathcal{P}} p_{k|k-1}\Big[hP_D G_z^{|W|}(p_z[g])\prod_{z'\in W}\frac{p_z(z')}{p_{FA}(z')}\Big]\Big)\end{aligned} \tag{77}$$

$$\begin{aligned}=&\Big(\prod_{z'\in Z} p_{FA}(z')\Big)\Big(G_{FA}^{(|Z|)}(p_{FA}[g])\\ &\times G_{k|k-1}\Big(p_{k|k-1}\big[h(1-P_D+P_D G_z(p_z[g]))\big]\Big)\\ &+G_{FA}(p_{FA}[g])\\ &\times\sum_{\mathcal{P}\angle Z} G_{k|k-1}^{(|\mathcal{P}|)}\Big(p_{k|k-1}\big[h(1-P_D+P_D G_z(p_z[g]))\big]\Big)\\ &\times\prod_{W\in\mathcal{P}} p_{k|k-1}\Big[hP_D G_z^{|W|}(p_z[g])\prod_{z'\in W}\frac{p_z(z')}{p_{FA}(z')}\Big]\\ &+\sum_{\substack{\mathcal{P}\angle Z\\ |\mathcal{P}|>1}}\sum_{W\in\mathcal{P}} G_{FA}^{(|W|)}(p_{FA}[g])\\ &\times G_{k|k-1}^{(|\mathcal{P}|-1)}\Big(p_{k|k-1}\big[h(1-P_D+P_D G_z(p_z[g]))\big]\Big)\\ &\times\prod_{W'\in\mathcal{P}-W} p_{k|k-1}\Big[hP_D G_z^{|W'|}(p_z[g])\prod_{z'\in W'}\frac{p_z(z')}{p_{FA}(z')}\Big]\Big)\end{aligned} \tag{78}$$

where we used the equality

$$\sum_{\substack{S\subseteq Z\\ S\neq Z}} f(Z-S)\sum_{\mathcal{P}\angle S} g(\mathcal{P})=\sum_{\substack{\mathcal{P}\angle Z\\ |\mathcal{P}|>1}}\sum_{W\in\mathcal{P}} f(W)g(\mathcal{P}-W) \tag{79}$$

while going from (77) to (78). Now, including the multiplication

$$G_{FA}^{(|Z|)}(p_{FA}[g])G_{k|k-1}\Big(p_{k|k-1}\big[h(1-P_D+P_D G_z(p_z[g]))\big]\Big) \tag{80}$$

into the summation $\sum_{\substack{\mathcal{P}\angle Z\\ |\mathcal{P}|>1}}$ as the case of $|\mathcal{P}|=1$, i.e., $\mathcal{P}=\{Z\}$ with the convention that $\prod_{W\in\phi}=1$, we get

$$\begin{aligned}\frac{\delta}{\delta Z}F[g,h]=&\Big(\prod_{z'\in Z} p_{FA}(z')\Big)\Big(G_{FA}(p_{FA}[g])\\ &\times\sum_{\mathcal{P}\angle Z} G_{k|k-1}^{(|\mathcal{P}|)}\Big(p_{k|k-1}\big[h(1-P_D+P_D G_z(p_z[g]))\big]\Big)\\ &\times\prod_{W\in\mathcal{P}} p_{k|k-1}\Big[hP_D G_z^{|W|}(p_z[g])\prod_{z'\in W}\frac{p_z(z')}{p_{FA}(z')}\Big]\\ &+\sum_{\mathcal{P}\angle Z}\sum_{W\in\mathcal{P}} G_{FA}^{(|W|)}(p_{FA}[g])\\ &\times G_{k|k-1}^{(|\mathcal{P}|-1)}\Big(p_{k|k-1}\big[h(1-P_D+P_D G_z(p_z[g]))\big]\Big)\\ &\times\prod_{W'\in\mathcal{P}-W} p_{k|k-1}\Big[hP_D G_z^{|W'|}(p_z[g])\prod_{z'\in W'}\frac{p_z(z')}{p_{FA}(z')}\Big]\Big)\end{aligned} \tag{81}$$

$$\begin{aligned}=&\Big(\prod_{z'\in Z} p_{FA}(z')\Big)\Big(\sum_{\mathcal{P}\angle Z} G_{FA}(p_{FA}[g])\\ &\times G_{k|k-1}^{(|\mathcal{P}|)}\Big(p_{k|k-1}\big[h(1-P_D+P_D G_z(p_z[g]))\big]\Big)\\ &\times\prod_{W\in\mathcal{P}} p_{k|k-1}\Big[hP_D G_z^{|W|}(p_z[g])\prod_{z'\in W}\frac{p_z(z')}{p_{FA}(z')}\Big]\\ &+\sum_{\mathcal{P}\angle Z}\sum_{W\in\mathcal{P}} G_{FA}^{(|W|)}(p_{FA}[g])\\ &\times G_{k|k-1}^{(|\mathcal{P}|-1)}\Big(p_{k|k-1}\big[h(1-P_D+P_D G_z(p_z[g]))\big]\Big)\\ &\times\prod_{W'\in\mathcal{P}-W} p_{k|k-1}\Big[hP_D G_z^{|W'|}(p_z[g])\prod_{z'\in W'}\frac{p_z(z')}{p_{FA}(z')}\Big]\Big)\end{aligned} \tag{82}$$

$$\begin{aligned}=&\Big(\prod_{z'\in Z} p_{FA}(z')\Big)\sum_{\mathcal{P}\angle Z}\Big(G_{FA}(p_{FA}[g])\\ &\times G_{k|k-1}^{(|\mathcal{P}|)}\Big(p_{k|k-1}\big[h(1-P_D+P_D G_z(p_z[g]))\big]\Big)\\ &\times\frac{1}{|\mathcal{P}|}\sum_{W\in\mathcal{P}} p_{k|k-1}\Big[hP_D G_z^{|W|}(p_z[g])\prod_{z'\in W}\frac{p_z(z')}{p_{FA}(z')}\Big]\\ &\times\prod_{W'\in\mathcal{P}-W} p_{k|k-1}\Big[hP_D G_z^{|W'|}(p_z[g])\prod_{z'\in W'}\frac{p_z(z')}{p_{FA}(z')}\Big]\\ &+\sum_{W\in\mathcal{P}} G_{FA}^{(|W|)}(p_{FA}[g])\\ &\times G_{k|k-1}^{(|\mathcal{P}|-1)}\Big(p_{k|k-1}\big[h(1-P_D+P_D G_z(p_z[g]))\big]\Big)\\ &\times\prod_{W'\in\mathcal{P}-W} p_{k|k-1}\Big[hP_D G_z^{|W'|}(p_z[g])\prod_{z'\in W'}\frac{p_z(z')}{p_{FA}(z')}\Big]\Big)\end{aligned} \tag{83}$$

$$
\begin{aligned}
&= \left(\prod_{z'\in Z} p_{FA}(z')\right) \sum_{\mathcal{P}\angle Z} \sum_{W\in\mathcal{P}} \Bigg(G_{FA}(p_{FA}[g]) \\
&\times G_{k|k-1}^{(|\mathcal{P}|)}\Big(p_{k|k-1}\big[h(1-P_D+P_D G_z(p_z[g]))\big]\Big) \\
&\times \frac{1}{|\mathcal{P}|} p_{k|k-1}\Big[hP_D G_z^{|W|}(p_z[g]) \prod_{z'\in W} \frac{p_z(z')}{p_{FA}(z')}\Big] \\
&+ G_{FA}^{(|W|)}(p_{FA}[g]) \\
&\times G_{k|k-1}^{(|\mathcal{P}|-1)}\Big(p_{k|k-1}\big[h(1-P_D+P_D G_z(p_z[g]))\big]\Big)\Bigg) \\
&\times \prod_{W'\in\mathcal{P}-W} p_{k|k-1}\Big[hP_D G_z^{|W'|}(p_z[g]) \prod_{z'\in W'} \frac{p_z(z')}{p_{FA}(z')}\Big]
\end{aligned} \tag{84}
$$

which is the result of Theorem 3.2 when we substitute the terms $\eta_W[g,h]$ defined in (24). Proof is complete. □

Appendix C
Proof of Theorem 3.3

Taking the derivative of both sides of (27), we get

$$
\begin{aligned}
&\frac{\delta}{\delta x}\frac{\delta}{\delta Z} F[0,h] = \left(\prod_{z'\in Z} p_{FA}(z')\right) p_{k|k-1}(x) \\
&\times \sum_{\mathcal{P}\angle Z} \sum_{W\in\mathcal{P}} \left(\prod_{W'\in\mathcal{P}-W} \eta_{W'}[0,h]\right) \\
&\times \Bigg(G_{FA}(0) G_{k|k-1}^{(|\mathcal{P}|+1)}(\rho[h])(1-P_D(x)+P_D(x)G_z(0)) \\
&\times \frac{\eta_W[0,h]}{|\mathcal{P}|} + G_{FA}(0) G_{k|k-1}^{(|\mathcal{P}|)}(\rho[h]) \frac{h(x)P_D(x)G_z^{|W|}(0)}{|\mathcal{P}|} \\
&\times \prod_{z'\in W} \frac{p_z(z'|x)}{p_{FA}(z')} + G_{FA}^{(|W|)}(0) G_{k|k-1}^{(|\mathcal{P}|)}(\rho[h]) \\
&\times (1-P_D(x)+P_D(x)G_z(0)) \\
&+ \Big(G_{FA}(0) G_{k|k-1}^{(|\mathcal{P}|)}(\rho[h]) \frac{\eta_W[0,h]}{|\mathcal{P}|} \\
&+ G_{FA}^{(|W|)}(0) G_{k|k-1}^{(|\mathcal{P}|-1)}(\rho[h]) \Big) \\
&\times \sum_{W'\in\mathcal{P}-W} \frac{h(x)P_D(x)G_z^{|W'|}(0)}{\eta_{W'}[0,h]} \prod_{z'\in W'} \frac{p_z(z'|x)}{p_{FA}(z')} \Bigg)
\end{aligned} \tag{85}
$$

$$
\begin{aligned}
&= \left(\prod_{z'\in Z} p_{FA}(z')\right) p_{k|k-1}(x) \\
&\times \sum_{\mathcal{P}\angle Z} \sum_{W\in\mathcal{P}} \left(\prod_{W'\in\mathcal{P}-W} \eta_{W'}[0,h]\right) \\
&\times \Bigg(\Big(G_{FA}(0) G_{k|k-1}^{(|\mathcal{P}|+1)}(\rho[h]) \frac{\eta_W[0,h]}{|\mathcal{P}|} \\
&+ G_{FA}^{(|W|)}(0) G_{k|k-1}^{(|\mathcal{P}|)}(\rho[h]) \Big)(1-P_D(x)+P_D(x)G_z(0)) \\
&+ G_{FA}(0) G_{k|k-1}^{(|\mathcal{P}|)}(\rho[h]) \frac{h(x)P_D(x)G_z^{|W|}(0)}{|\mathcal{P}|} \prod_{z'\in W} \frac{p_z(z'|x)}{p_{FA}(z')} \\
&+ \Big(G_{FA}(0) G_{k|k-1}^{(|\mathcal{P}|)}(\rho[h]) \frac{\eta_W[0,h]}{|\mathcal{P}|} \\
&+ G_{FA}^{(|W|)}(0) G_{k|k-1}^{(|\mathcal{P}|-1)}(\rho) \Big) \\
&\times \sum_{W'\in\mathcal{P}-W} \frac{h(x)P_D(x)G_z^{|W'|}(0)}{\eta_{W'}[0,h]} \prod_{z'\in W'} \frac{p_z(z'|x)}{p_{FA}(z')} \Bigg)
\end{aligned} \tag{86}
$$

which is the result of Theorem 3.3. The proof is complete. □

Appendix D
Reducing ETT-CPHD to ETT-PHD

When one constrains all the processes to Poisson, we get the following identities.

$$
\begin{aligned}
G_{FA}^{(n)}(x) &= \lambda^n G_{FA}^{(n)}(x) && (87)\\
G_z^{(n)}(x) &= \tau^n G_z^{(n)}(x) && (88)\\
G_{k|k-1}^{(n)}(x) &= N_{k|k-1}^n G_{k|k-1}^{(n)}(x) && (89)
\end{aligned}
$$

where λ and τ are the expected number of false alarms and measurements from a single target. $N_{k|k-1}$ is the predicted expected number of targets i.e., $N_k|k-1 = \int D_{k|k-1}(x)\,\mathrm{d}x$. We are first going to examine the constant κ defined in (37). For this purpose, we first write

$$
\begin{aligned}
\beta_{\mathcal{P},W} \triangleq& G_{FA}(0) G_{k|k-1}(\rho[1]) \Big(N_{k|k-1}^{|\mathcal{P}|} \frac{\eta_W[0,1]}{|\mathcal{P}|} \\
&+ \lambda^{|W|} N_{k|k-1}^{|\mathcal{P}|-1} \Big);
\end{aligned} \tag{90}
$$

$$
\begin{aligned}
\gamma_{\mathcal{P},W} \triangleq& G_{FA}(0) G_{k|k-1}(\rho[1]) \Big(N_{k|k-1}^{|\mathcal{P}|+1} \frac{\eta_W[0,1]}{|\mathcal{P}|} \\
&+ \lambda^{|W|} N_{k|k-1}^{|\mathcal{P}|} \Big).
\end{aligned} \tag{91}
$$

Now, substituting these into (37), we get

$$
\kappa = \frac{\sum_{\mathcal{P}\angle Z}\sum_{W\in\mathcal{P}} \alpha_{\mathcal{P},W} \times \Big(N_{k|k-1}^{|\mathcal{P}|+1} \frac{\eta_W[0,1]}{|\mathcal{P}|} + \lambda^{|W|} N_{k|k-1}^{|\mathcal{P}|} \Big)}{\sum_{\mathcal{P}\angle Z}\sum_{W\in\mathcal{P}} \alpha_{\mathcal{P},W} \times \Big(N_{k|k-1}^{|\mathcal{P}|} \frac{\eta_W[0,1]}{|\mathcal{P}|} + \lambda^{|W|} N_{k|k-1}^{|\mathcal{P}|-1} \Big)} = N_{k|k-1}. \tag{92}
$$

We here define the modified versions of $\eta_W[0,1]$, and $\alpha_{\mathcal{P},W}$ as

$$
\begin{aligned}
\bar{\eta}_W[0,1] \triangleq& \frac{N_{k|k-1}\eta_W[0,1]}{\lambda^{|W|}} && (93)\\
=& D_{k|k-1}\Big[P_D G_z^{(|W|)}(0) \prod_{z'\in W} \frac{p_z(z')}{\lambda p_{FA}(z')}\Big] && (94)\\
\bar{\alpha}_{\mathcal{P},W} \triangleq& N_{k|k-1}^{|P|-1} \lambda^{|W|} \alpha_{\mathcal{P},W}[0,1] && (95)\\
=& \lambda^{|Z|} \prod_{W'\in\mathcal{P}-W} \bar{\eta}_{W'}[0,1]. && (96)
\end{aligned}
$$

With these definitions $\beta_{\mathcal{P},W}$ and the multiplication $\alpha_{\mathcal{P},W}\beta_{\mathcal{P},W}$ become

$$\beta_{\mathcal{P},W} \triangleq G_{FA}(0)G_{k|k-1}(\rho[1])N_{k|k-1}^{|\mathcal{P}|-1}\lambda^{|W|} \times \left(\frac{\bar{\eta}_W[0,1]}{|\mathcal{P}|}+1\right) \quad (97)$$

$$\alpha_{\mathcal{P},W}\beta_{\mathcal{P},W} = G_{FA}(0)G_{k|k-1}(\rho[1])\bar{\alpha}_{\mathcal{P},W} \times \left(\frac{\bar{\eta}_W[0,1]}{|\mathcal{P}|}+1\right). \quad (98)$$

Using these, we can write $\sigma_{\mathcal{P},W}$ as

$$\sigma_{\mathcal{P},W} \triangleq G_z^{|W|}(0)G_{FA}(0)G_{k|k-1}(\rho[1])N_{k|k-1}\lambda^{-|W|}\left(\frac{\bar{\alpha}_{\mathcal{P},W}}{|\mathcal{P}|} + \frac{\sum_{W'\in\mathcal{P}-W}\bar{\alpha}_{\mathcal{P},W'}\left(\frac{\bar{\eta}_{W'}[0,1]}{|\mathcal{P}|}+1\right)}{\bar{\eta}_W[0,1]}\right). \quad (99)$$

After defining the quantity $\zeta_{\mathcal{P}}$ as

$$\zeta_{\mathcal{P}} \triangleq \prod_{W\in\mathcal{P}}\bar{\eta}_W[0,1] = \bar{\alpha}_{\mathcal{P},W}\bar{\eta}_W[0,1], \quad (100)$$

we can turn (99) into

$$\begin{aligned}\sigma_{\mathcal{P},W} \triangleq& G_z^{|W|}(0)G_{FA}(0)G_{k|k-1}(\rho[1])N_{k|k-1}\lambda^{-|W|} \\ &\times\frac{\zeta_{\mathcal{P}}+|\mathcal{P}|\sum_{W'\in\mathcal{P}-W}\bar{\alpha}_{\mathcal{P},W'}\left(\frac{\bar{\eta}_{W'}[0,1]}{|\mathcal{P}|}+1\right)}{|\mathcal{P}|\bar{\eta}_W[0,1]} \quad (101)\\ =& G_z^{|W|}(0)G_{FA}(0)G_{k|k-1}(\rho[1])N_{k|k-1}\lambda^{-|W|} \\ &\times\frac{|\mathcal{P}|\zeta_{\mathcal{P}}+|\mathcal{P}|\sum_{W'\in\mathcal{P}-W}\zeta_{\mathcal{P}-W'}}{|\mathcal{P}|\bar{\eta}_W[0,1]} \quad (102)\\ =& G_z^{|W|}(0)G_{FA}(0)G_{k|k-1}(\rho[1])N_{k|k-1}\lambda^{-|W|} \\ &\times\frac{\zeta_{\mathcal{P}}+\sum_{W'\in\mathcal{P}-W}\zeta_{\mathcal{P}-W'}}{\bar{\eta}_W[0,1]} \quad (103)\\ =& G_z^{|W|}(0)G_{FA}(0)G_{k|k-1}(\rho[1])N_{k|k-1}\lambda^{-|W|} \\ &\times\left(\zeta_{\mathcal{P}-W}+\sum_{W'\in\mathcal{P}-W}\zeta_{\mathcal{P}-W-W'}\right) \quad (104)\\ =& e^{-\tau}\tau^{|W|}G_{FA}(0)G_{k|k-1}(\rho[1])N_{k|k-1}\lambda^{-|W|} \\ &\times\left(\zeta_{\mathcal{P}-W}+\sum_{W'\in\mathcal{P}-W}\zeta_{\mathcal{P}-W-W'}\right). \quad (105)\end{aligned}$$

We can write by using (105) and (98)

$$\begin{aligned}&\frac{\sigma_{\mathcal{P},W}}{\sum_{\mathcal{P}\angle Z}\sum_{W\in\mathcal{P}}\alpha_{\mathcal{P},W}\beta_{\mathcal{P},W}} \\ &= \frac{\left(\begin{array}{c}N_{k|k-1}e^{-\tau}\tau^{|W|}\lambda^{-|W|}\\ \times\left(\zeta_{\mathcal{P}-W}+\sum_{W'\in\mathcal{P}-W}\zeta_{\mathcal{P}-W-W'}\right)\end{array}\right)}{\sum_{\mathcal{P}\angle Z}\sum_{W\in\mathcal{P}}\bar{\alpha}_{\mathcal{P},W}\left(\frac{\bar{\eta}_W[0,1]}{|\mathcal{P}|}+1\right)} \quad (106)\\ &= N_{k|k-1}\frac{\left(\begin{array}{c}e^{-\tau}\tau^{|W|}\lambda^{-|W|}\\ \times\left(\zeta_{\mathcal{P}-W}+\sum_{W'\in\mathcal{P}-W}\zeta_{\mathcal{P}-W-W'}\right)\end{array}\right)}{\sum_{\mathcal{P}\angle Z}\left(\zeta_{\mathcal{P}}+\sum_{W\in\mathcal{P}}\zeta_{\mathcal{P}-W}\right)}. \quad (107)\end{aligned}$$

Substituting (107) into the update equation (39) for $D_{k|k-1}(\cdot)$, we get

$$\begin{aligned}D_{k|k}(x) =& (1-P_D(x)+P_D(x)G_z(0))D_{k|k-1}(x) \\ &+ e^{-\tau}\frac{\left(\begin{array}{c}\sum_{\mathcal{P}\angle Z}\sum_{W\in\mathcal{P}}\left(\zeta_{\mathcal{P}-W}\right.\\ \left.+\sum_{W'\in\mathcal{P}-W}\zeta_{\mathcal{P}-W-W'}\right)\\ \times\tau^{|W|}\prod_{z'\in W}\frac{p_z(z'|x)}{\lambda p_{FA}(z')}P_D(x)\end{array}\right)}{\sum_{\mathcal{P}\angle Z}\left(\zeta_{\mathcal{P}}+\sum_{W\in\mathcal{P}}\zeta_{\mathcal{P}-W}\right)}D_{k|k-1}(x).\end{aligned} \quad (108)$$

- First, we examine the term $\sum_{\mathcal{P}\angle Z}\left(\zeta_{\mathcal{P}}+\sum_{W\in\mathcal{P}}\zeta_{\mathcal{P}-W}\right)$ in the denominator of (108) as follows.

$$\begin{aligned}\sum_{\mathcal{P}\angle Z}\left(\zeta_{\mathcal{P}}+\sum_{W\in\mathcal{P}}\zeta_{\mathcal{P}-W}\right) &= \sum_{\mathcal{P}\angle Z}\zeta_{\mathcal{P}}+\sum_{\mathcal{P}\angle Z}\sum_{W\in\mathcal{P}}\zeta_{\mathcal{P}-W} \quad (109)\\ &= \sum_{\mathcal{P}\angle Z}\zeta_{\mathcal{P}}+\sum_{\substack{\mathcal{P}\angle Z\\|\mathcal{P}|>1}}\sum_{W\in\mathcal{P}}\zeta_{\mathcal{P}-W}+1. \quad (110)\end{aligned}$$

If we use the identity (79) on the second term in the summation on the right hand side of (110), we obtain

$$\begin{aligned}\sum_{\mathcal{P}\angle Z}\left(\zeta_{\mathcal{P}}+\sum_{W\in\mathcal{P}}\zeta_{\mathcal{P}-W}\right) &= \sum_{\mathcal{P}\angle Z}\zeta_{\mathcal{P}}+\sum_{\substack{S\subset Z\\S\neq Z}}\sum_{\mathcal{P}\angle S}\zeta_{\mathcal{P}}+1 \quad (111)\\ &= \sum_{S\subset Z}\sum_{\mathcal{P}\angle S}\zeta_{\mathcal{P}}+1 \quad (112)\\ &= \sum_{\mathcal{P}\angle Z}\prod_{W\in\mathcal{P}}d_W \quad (113)\end{aligned}$$

where

$$d_W = \delta_{|W|=1}+\bar{\eta}_W[0,1]. \quad (114)$$

- We examine the numerator term in (108) below.

$$\sum_{\mathcal{P}\angle Z}\sum_{W\in\mathcal{P}}g_{\text{num}}(\mathcal{P}-W)f_{\text{num}}(W) \quad (115)$$

where we defined

$$f_{\text{num}}(W) \triangleq \tau^{|W|}\prod_{z'\in W}\frac{p_z(z'|x)}{\lambda p_{FA}(z')}P_D(x) \quad (116)$$

$$g_{\text{num}}(\mathcal{P}) \triangleq \left(\zeta_{\mathcal{P}}+\sum_{W'\in\mathcal{P}}\zeta_{\mathcal{P}-W'}\right). \quad (117)$$

Separating the case $\mathcal{P} = \{Z\}$ from the summation in (115), we get

$$\begin{aligned}\sum_{\mathcal{P}\angle Z}\sum_{W\in\mathcal{P}}g_{\text{num}}(\mathcal{P}-W)f_{\text{num}}(W) =& f_{\text{num}}(Z) \\ &+\sum_{\substack{\mathcal{P}\angle Z\\|\mathcal{P}|>1}}\sum_{W\in\mathcal{P}}g_{\text{num}}(\mathcal{P}-W)f_{\text{num}}(W) \quad (118)\\ =& f_{\text{num}}(Z)+\sum_{\substack{S\subset Z\\S\neq Z}}f_{\text{num}}(S-W)\sum_{\mathcal{P}\angle S}g_{\text{num}}(\mathcal{P}). \quad (119)\end{aligned}$$

where we used the identity (79). Now realizing that we calculated the term $\sum_{\mathcal{P}\angle Z} g_{\text{num}}(\mathcal{P})$ as in (113), we can substitute this into (119) as follows.

$$\sum_{\mathcal{P}\angle Z}\sum_{W\in\mathcal{P}} g_{\text{num}}(\mathcal{P}-W)f_{\text{num}}(W) = f_{\text{num}}(Z) + \sum_{\substack{S\subset Z\\ S\neq Z}} f_{\text{num}}(S-W)\sum_{\mathcal{P}\angle S}\prod_{W\in\mathcal{P}} d_W \tag{120}$$

Resorting to the identity (79) once again, we get

$$\sum_{\mathcal{P}\angle Z}\sum_{W\in\mathcal{P}} g_{\text{num}}(\mathcal{P}-W)f_{\text{num}}(W) = f_{\text{num}}(Z) + \sum_{\substack{\mathcal{P}\angle Z\\ |\mathcal{P}|>1}}\sum_{W\in\mathcal{P}} f_{\text{num}}(W)\prod_{W'\in\mathcal{P}-W} d_{W'} \tag{121}$$

$$= \sum_{\mathcal{P}\angle Z}\sum_{W\in\mathcal{P}} f_{\text{num}}(W)\prod_{W'\in\mathcal{P}-W} d_{W'}. \tag{122}$$

When we substitute the results (113) and (122) back into (108), we obtain

$$D_{k|k}(x) = \Bigg((1 - P_D(x) + P_D(x)G_z(0)) + e^{-\tau}\sum_{\mathcal{P}\angle Z}\omega_{\mathcal{P}}\sum_{W\in\mathcal{P}}\frac{\tau^{|W|}}{d_W}\prod_{z'\in W}\frac{p_z(z'|x)}{\lambda p_{FA}(z')}P_D(x)\Bigg) \times D_{k|k-1}(x) \tag{123}$$

where

$$\omega_{\mathcal{P}} \triangleq \frac{\prod_{W\in\mathcal{P}} d_W}{\sum_{\mathcal{P}\angle Z}\prod_{W\in\mathcal{P}} d_W} \tag{124}$$

which is the same PHD update equation as in [6, equation (5)]. Moreover, the terms d_W and $\omega_{\mathcal{P}}$ are the same as those defined in [6, equations (7) and (6) respectively].

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Extended Target Tracking with a Cardinalized Probability Hypothesis Density Filter

Författare
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Sammanfattning
Abstract

This technical report presents a cardinalized probability hypothesis density (CPHD) filter for extended targets that can result in multiple measurements at each scan. The probability hypothesis density (PHD) filter for such targets has already been derived by Mahler and a Gaussian mixture implementation has been proposed recently. This work relaxes the Poisson assumptions of the extended target PHD filter in target and measurement numbers to achieve better estimation performance. A Gaussian mixture implementation is described. The early results using real data from a laser sensor confirm that the sensitivity of the number of targets in the extended target PHD filter can be avoided with the added flexibility of the extended target CPHD filter.

Nyckelord
Keywords

Multiple target tracking, extended targets, random sets, probability hypothesis density, car-