Analysis of Methods for Multivariable Frequency Response Function Estimation in Closed Loop

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Abstract

Estimation methods for the multivariable frequency response function are analyzed, both in open and closed loop. Approximate bias and covariance expressions are derived for the closed loop case. The usefulness of these expressions is illustrated in simulations of an industrial robot where three different estimators are compared. The choice of estimator depends on the signal-to-noise ratio as well as the measurement setup and a bias-variance trade-off.

Keywords: Nonparametric identification, Multivariable systems, Closed Loop

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Abstract—Estimation methods for the multivariable frequency response function are analyzed, both in open and closed loop. Approximate bias and covariance expressions are derived for the closed loop case. The usefulness of these expressions is illustrated in simulations of an industrial robot where three different estimators are compared. The choice of estimator depends on the signal-to-noise ratio as well as the measurement setup and a bias-variance trade-off.

I. INTRODUCTION

This paper mainly studies the properties of a frequency response function (FRF) estimation method for multivariable systems, described in [1], for both open loop and closed loop data. It is known, see for example [2], that this type of frequency domain method will give biased estimates when using closed loop data, but there are practical reasons why it anyway is motivated to study this problem. First, in many applications it is more or less inevitable to use feedback control during the data collection, and, second, the frequency domain identification method has appealing properties in its simplicity.

Previous studies of the performance of the method are presented in, *e.g.*, [1] and [3] for open loop data and [4] contains some early work for closed loop data. In this paper, the FRF estimation method will be analyzed for both open loop and closed loop data. The open loop covariance expressions from the two papers [1] and [3] will here be combined. For closed loop data, an asymptotic expression for the bias is derived together with approximate covariance expressions. For numerical illustration, a simulation model of an industrial robot will be used. The robot application is interesting since it gives many challenging problems for system identification methods, such as a multivariable non-linear system, oscillatory behavior, and data collection under feedback control. An overview of identification in robotics can be found in [5]. See also [6], [7], [8] for examples where FRF estimates are used for the identification of parametric robot models. The simulation results are evaluated using the derived bias and variance expressions. Corresponding results using experimental data from real robots are presented in, *e.g.*, [9] and [10].

II. THE IDENTIFICATION METHOD

The setup considered in this paper is given by

$$y(t) = G(q)(u(t) + v_u(t)) + v_y(t), \quad (1)$$

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where $G(q)$ is the $n_y \times n_u$ multivariable discrete-time transfer operator, with q being the shift operator, and $v_u(t)$ and $v_y(t)$ are noise on the input and output, respectively. The input and output signals, $u(t) \in \mathbb{R}^{n_u}$ and $y(t) \in \mathbb{R}^{n_y}$, are measured at time instants $t_n = nT_s$, $n = 1, 2, \dots, N$, with sample time T_s . Equation (1) can actually be transformed to an output noise case,

$$y(t) = G(q)u(t) + v(t), \quad v(t) = G(q)v_u(t) + v_y(t). \quad (2)$$

To avoid leakage effects in the discrete Fourier transform (DFT), which is used by the identification method, the input signal, $u(t)$, is assumed to be N_P -periodic and an integer number of periods, P , of the steady state response is collected, giving $N = PN_P$ samples for each experiment. Consider now the DFTs of the measured input and output signals

$$U(\omega_k) = \frac{1}{\sqrt{N}} \sum_{n=1}^N u(nT_s) e^{j\omega_k nT_s},$$

$$Y(\omega_k) = \frac{1}{\sqrt{N}} \sum_{n=1}^N y(nT_s) e^{j\omega_k nT_s},$$

where only the N_P frequencies $\omega_k = k \frac{2\pi}{N_P T_s}$, $k = 1, 2, \dots, N_P$ are considered. Given the periodic data, the following linear mapping will hold exactly in the noise free case

$$Y(\omega_k) = G(e^{j\omega_k T_s})U(\omega_k), \quad (3)$$

where $G(e^{j\omega_k T_s}) \in \mathbb{C}^{n_y \times n_u}$ is the FRF. To be able to extract $G(e^{j\omega_k T_s})$ from data, at least n_u different experiments are needed. The data vectors from $n_e \geq n_u$ different experiments will be collected into matrices (bold face in the sequel) where each column corresponds to one experiment. The input-output relation can then be written as

$$\mathbf{Y}(\omega_k) = G(e^{j\omega_k T_s})\mathbf{U}(\omega_k), \quad (4)$$

where $\mathbf{U}(\omega_k) \in \mathbb{C}^{n_u \times n_e}$ and $\mathbf{Y}(\omega_k) \in \mathbb{C}^{n_y \times n_e}$. If $\mathbf{U}(\omega_k)$ has rank n_u , an estimate of $G(e^{j\omega_k T_s})$ can be formed using the H_1 estimator (see also (10))

$$\hat{G}(e^{j\omega_k T_s}) = \mathbf{Y}(\omega_k)\mathbf{U}^\dagger(\omega_k), \quad (5)$$

where $\mathbf{U}^\dagger(\omega_k)$ is the pseudo-inverse $\mathbf{U}^\dagger = \mathbf{U}^H(\mathbf{U}\mathbf{U}^H)^{-1}$ and $(\cdot)^H$ denotes complex conjugate transpose. When the measurements are corrupted by noise, (4) changes to

$$\mathbf{Y}(\omega_k) = G(e^{j\omega_k T_s})\mathbf{U}(\omega_k) + \mathbf{V}(\omega_k), \quad (6a)$$

$$\mathbf{V}(\omega_k) = \mathbf{V}_y(\omega_k) + G(e^{j\omega_k T_s})\mathbf{V}_u(\omega_k). \quad (6b)$$

The H_1 estimator (5) can still be used, but the estimate will contain errors due to the noise.

In general the choice of excitation signal offers a large freedom in terms of frequency contents, magnitude, etc, as long as the matrix $\mathbf{U}(\omega_k)$ has rank n_u . In this paper, the orthogonal random phase multisine signal will be used. This signal has been suggested in [3] and [11] to minimize the variance ($\det \sigma_{\hat{G}}^2$, cf. (16)) given certain amplitude constraints for the input signal. Assuming $n_e = M \cdot n_u$, $\mathbf{U}(\omega_k)$ is partitioned in M blocks of size $n_u \times n_u$ as

$$\mathbf{U}(\omega_k) = [\mathbf{U}^{(1)}(\omega_k) \quad \dots \quad \mathbf{U}^{(M)}(\omega_k)], \quad (7a)$$

$$\mathbf{U}^{(m)}(\omega_k) = \mathbf{U}_{\text{diag}}^{(m)}(\omega_k) \mathbf{T}, \quad m = 1, \dots, M, \quad (7b)$$

with $\mathbf{U}_{\text{diag}}^{(m)}(\omega_k) = \text{diag} \left\{ U_l^{(m)}(\omega_k) \right\}_{l=1}^{n_u}$ a diagonal matrix where each $U_l^{(m)}(\omega_k)$ is a random phase multisine signal, and $\mathbf{T}$ is an orthogonal matrix. A scalar random phase multisine signal $u(t)$ can be written as

$$u(t) = \sum_{k=1}^{n_f} A_k \sin(\omega_k t + \phi_k), \quad (8)$$

with amplitudes A_k , frequencies ω_k chosen from the grid $\left\{ \frac{2\pi l}{N_P T_s}, l = 1, \dots, N_P/2 - 1 \right\}$, and random phases ϕ_k uniformly distributed in $[0, 2\pi)$. The optimal matrix $\mathbf{T}$, with amplitude constraints $|\mathbf{T}_{il}| \leq 1$, is given by [11]

$$\mathbf{T}_{il} = e^{\frac{2\pi j}{n_u} (i-1)(l-1)}. \quad (9)$$

For the single input, single output (SISO) case, a number of different FRF estimators have been suggested in the literature (see, e.g., [12] and [13]), which all have different properties regarding bias (for the output noise case only in closed loop) and variance. These estimators can often be generalized to the multiple input, multiple output (MIMO) case. If, for example, (7a) is used, (5) can be rewritten as

$$\hat{G} = \left[\frac{1}{M} \sum_{m=1}^M \mathbf{Y}^{(m)} \mathbf{U}^{(m)H} \right] \left[\frac{1}{M} \sum_{m=1}^M \mathbf{U}^{(m)} \mathbf{U}^{(m)H} \right]^{-1}. \quad (10)$$

(For notational simplicity, the frequency argument will be omitted when not explicitly needed.) This is why (5) and (10) will be called the H_1 estimator (cf. [12]). Another useful estimator is the *arithmetic mean estimator*

$$\hat{G}^{\text{ARI}} = \frac{1}{M} \sum_{m=1}^M \hat{G}^{(m)} = \frac{1}{M} \sum_{m=1}^M \mathbf{Y}^{(m)} [\mathbf{U}^{(m)}]^{-1}. \quad (11)$$

In case (7) with (9) is used as input with the same amplitude distribution over the M blocks ($U_l^{(m)} = U_l$, $m = 1, \dots, M$), the H_1 and ARI estimators, (10) and (11), will give the same estimate. The H_1 estimator is more robust than the ARI estimator if certain blocks have poor SNR, but will typically give a larger bias for moderate SNR's. If the reference signal is measured, an asymptotically ($M \rightarrow \infty$) unbiased estimator has been proposed in [13], which can be generalized to the

MIMO case as

$$\hat{G}^{\text{JIO}} = \left[\frac{1}{M} \sum_{m=1}^M \mathbf{Y}^{(m)} \mathbf{R}^{(m)H} \right] \left[\frac{1}{M} \sum_{m=1}^M \mathbf{U}^{(m)} \mathbf{R}^{(m)H} \right]^{-1}, \quad (12)$$

where $\mathbf{R}$ is the reference DFT matrix (cf. (21)) and JIO stands for *joint input-output estimator*. If the same excitation is used in all blocks and the measurements are synchronized, then (12) reduces to the *errors-in-variables estimator* [12]

$$\hat{G}^{\text{EIV}} = \left[\frac{1}{M} \sum_{m=1}^M \mathbf{Y}^{(m)} \right] \left[\frac{1}{M} \sum_{m=1}^M \mathbf{U}^{(m)} \right]^{-1}. \quad (13)$$

III. OPEN LOOP ERROR ANALYSIS

In this section, the H_1 estimator (5) will be analyzed for open loop data. This analysis is closely related to the work in [1] and [3]. In [1] the analysis is conducted within an errors-in-variables¹ framework, but the same covariance expression is obtained as for the the output noise case in this paper (see Theorem 1). In [3], covariance expressions are derived when the orthogonal multisine signal (7) is used as input. The MIMO system is there viewed as n_y separate MISO systems with output noise. Implicitly, the output noise is assumed to be independent over the different outputs so expressions for the covariance between different rows in $\hat{G}$ are not presented. In this paper, the MIMO system will be considered without such noise assumption, which can be seen as applying (7) to Theorem 1. In addition, in Section IV the results will be extended to the closed loop case.

A. Noise Assumptions

For the analysis, we will assume the following noise properties.

Assumption 1: For each of the n_e experiments, the DFTs of the input and output noise, $V_u(\omega_k)$ and $V_y(\omega_k)$ satisfy

$$\begin{aligned} \mathbb{E} V_y(\omega_k) &= 0, \quad \mathbb{E} V_u(\omega_k) = 0, \\ \mathbb{E} V_y(\omega_k) V_y^T(\omega_k) &= 0, \quad \mathbb{E} V_y(\omega_k) V_y^H(\omega_k) = \sigma_{V_y}^2(\omega_k), \\ \mathbb{E} V_u(\omega_k) V_u^T(\omega_k) &= 0, \quad \mathbb{E} V_u(\omega_k) V_u^H(\omega_k) = \sigma_{V_u}^2(\omega_k), \\ \mathbb{E} V_u(\omega_k) V_y^T(\omega_k) &= \mathbb{E} V_u(\omega_k) V_y^H(\omega_k) = 0, \end{aligned}$$

for $k = 1, \dots, n_f$, which means that $V_u(\omega_k)$ and $V_y(\omega_k)$ are independent and circular complex. In addition, it is assumed that they are independent and identically distributed over the n_e different experiments.

This assumption is common and justifiable in most practical circumstances² and enables us to obtain the following result for the covariance of $\mathbf{V}(\omega_k)$.

Lemma 1: Consider the DFT matrix $\mathbf{V}(\omega_k) \in \mathbb{C}^{n_y \times n_e}$ in (6). Under Assumption 1, the covariance $\sigma_{\mathbf{V}}^2(\omega_k)$ is given by

$$\sigma_{\mathbf{V}}^2(\omega_k) = \mathbb{E} \text{vec } \mathbf{V}(\omega_k) [\text{vec } \mathbf{V}(\omega_k)]^H = \mathbf{I} \otimes \sigma_V^2(\omega_k), \quad (14)$$

where $\otimes$ is the Kronecker product and

$$\sigma_V^2(\omega_k) = \sigma_{V_y}^2(\omega_k) + G(e^{j\omega_k T_s}) \sigma_{V_u}^2(\omega_k) G^H(e^{j\omega_k T_s}). \quad (15)$$

¹Measurements $y(t)$ and $u(t) + v_u(t)$ with $y(t) = G(q)u(t) + v_y(t)$.

²For example, the DFT of filtered white noise is asymptotically circular complex when the number of samples, $N \rightarrow \infty$, see [14, Theorem 14.25].

Proof: Let $\mathbf{V}_i$ denote column i in $\mathbf{V}$. $\sigma_{\mathbf{V}}^2$ will then be a block matrix with blocks $\mathbf{E} \mathbf{V}_i \mathbf{V}_j^H \in \mathbb{C}^{n_y \times n_y}$. Expanding this expression gives

$$\begin{aligned} \mathbf{E} \mathbf{V}_i \mathbf{V}_j^H &= \mathbf{E} \mathbf{V}_{y,i} \mathbf{V}_{y,j}^H + \mathbf{E} \mathbf{V}_{y,i} \mathbf{V}_{u,j}^H G^H + \\ &\quad + \mathbf{E} G \mathbf{V}_{u,i} \mathbf{V}_{y,j}^H + \mathbf{E} G \mathbf{V}_{u,i} \mathbf{V}_{u,j}^H G^H \\ &= \begin{cases} \sigma_{V_y}^2 + G \sigma_{V_u}^2 G^H & i = j \\ 0 & i \neq j \end{cases}, \end{aligned}$$

where the last step follows from Assumption 1. $\sigma_{\mathbf{V}}^2$ will therefore be a block-diagonal matrix with $\sigma_{V_y}^2$ in all the diagonal blocks, which can be written as $I \otimes \sigma_{V_y}^2$. ■

B. Bias and Covariance

The bias and variance of the H_1 estimator (5) will now be derived in the following theorem. The same result (for the case $n_e = n_u$) is stated without proof in [1] as a generalization of [15, Theorem 8.2.5] to the errors-in-variables framework. Here, a proof will be given for the output noise case, see also [15, Theorem 8.2.5].

Theorem 1: Consider the H_1 estimator (5) and assume an open loop setup (6) where $\mathbf{U}(\omega_k)$ and $\mathbf{V}(\omega_k)$ are independent and the noise fulfills Assumption 1. Then, the estimate $\hat{G}(e^{j\omega_k T_s})$ will be unbiased and the covariance $\sigma_{\hat{G}}^2(\omega_k)$ is given by

$$\begin{aligned} \sigma_{\hat{G}}^2(\omega_k) &= \mathbf{E} \text{vec} \tilde{G}(e^{j\omega_k T_s}) [\text{vec} \tilde{G}(e^{j\omega_k T_s})]^H = \\ &= [\mathbf{U}(\omega_k) \mathbf{U}^H(\omega_k)]^{-T} \otimes \sigma_{V_y}^2(\omega_k), \end{aligned} \quad (16)$$

with $\tilde{G} = \hat{G} - G$ and $\sigma_{V_y}^2$ from (15).

Proof: The estimation error $\tilde{G}$ can be rewritten as

$$\tilde{G} = \hat{G} - G = (G\mathbf{U} + \mathbf{V})\mathbf{U}^\dagger - G = \mathbf{V}\mathbf{U}^\dagger, \quad (17)$$

and $\mathbf{E} \tilde{G} = 0$ follows since $\mathbf{E} \mathbf{V} = 0$ and $\mathbf{U}$ and $\mathbf{V}$ are independent. For the covariance, $\text{vec} \tilde{G}$ can be rewritten as³ $\text{vec} \tilde{G} = (\mathbf{U}^{\dagger T} \otimes I) \text{vec} \mathbf{V}$ by using $\text{vec}(ADB) = (B^T \otimes A) \text{vec} D$. Equation (16) then follows from

$$\begin{aligned} \sigma_{\hat{G}}^2 &= \mathbf{E} \text{vec} \tilde{G} [\text{vec} \tilde{G}]^H \\ &= \mathbf{E} [\mathbf{U}^{\dagger T} \otimes I] \text{vec} \mathbf{V} (\text{vec} \mathbf{V})^H [\mathbf{U}^{\dagger T} \otimes I]^H \\ &= [\mathbf{U}^{\dagger T} \otimes I] [I \otimes \sigma_{V_y}^2] [\overline{\mathbf{U}^\dagger} \otimes I] \\ &= [\mathbf{U}^{\dagger T} \overline{\mathbf{U}^\dagger}] \otimes \sigma_{V_y}^2 = [\mathbf{U}^{\dagger H} \mathbf{U}^\dagger]^T \otimes \sigma_{V_y}^2 \\ &= [(\mathbf{U}\mathbf{U}^H)^{-1} \mathbf{U}\mathbf{U}^H (\mathbf{U}\mathbf{U}^H)^{-1}]^T \otimes \sigma_{V_y}^2, \end{aligned}$$

where $(A \otimes B)(C \otimes D) = (AB) \otimes (CD)$ has been used twice in the third step. ■

Continuing the error analysis with the orthogonal multisine signal (7) gives the following results (cf. [3]).

Corollary 1: Using an orthogonal multisine signal (7) with $\mathbf{T}$ from (9) simplifies $\sigma_{\hat{G}}^2$ in Theorem 1 to

$$\sigma_{\hat{G}}^2 = \text{diag} \left\{ \frac{1}{n_u \sum_{m=1}^M |U_l^{(m)}|^2} \right\}_{l=1}^{n_u} \otimes \sigma_{V_y}^2, \quad (18)$$

³Using basic properties for the Kronecker product of complex matrices, see for example [14, p. 420].

and the variance of each element in $\hat{G}$ as

$$\sigma_{\hat{G}_{ij}}^2 = \frac{\sigma_{V_{y,ii}}^2}{n_u \sum_{m=1}^M |U_j^{(m)}|^2}. \quad (19)$$

Proof: Follows by noting that $\mathbf{T}\mathbf{T}^H = n_u I$ for (9) as

$$\begin{aligned} \mathbf{U}\mathbf{U}^H &= \sum_m \mathbf{U}^{(m)} \mathbf{U}^{(m)H} = \sum_m \mathbf{U}_{\text{diag}}^{(m)} \mathbf{T} \mathbf{T}^H \mathbf{U}_{\text{diag}}^{(m)H} \\ &= n_u \sum_m \mathbf{U}_{\text{diag}}^{(m)} \mathbf{U}_{\text{diag}}^{(m)H} = n_u \sum_m \text{diag}\{|U_l^{(m)}|^2\}_{l=1}^{n_u}. \end{aligned}$$

This means that different columns in $\hat{G}$ are uncorrelated and that the covariance for a certain column is inversely proportional to the total input power in that particular input channel. Note that if another type of excitation signal $\mathbf{U}$ is used, without the property that $\mathbf{U}\mathbf{U}^H$ is diagonal, then the columns in $\hat{G}$ will be correlated. The variance (19) is given by a noise-to-signal ratio, where $\sigma_{V_{y,ii}}^2$ is the noise variance in output i and the denominator is the total power for input j during the n_e experiments. ■

IV. CLOSED LOOP ERROR ANALYSIS

In this paper the aim is to also analyze the properties of the identification method in a closed loop setup. Therefore, let the input be given by

$$u(t) = F(q)(r(t) - y(t)) \quad (20)$$

where $F(q)$ is the controller and $r(t) \in \mathbb{R}^{n_y}$ is the reference signal. For simplicity, $n_y = n_u$ will be assumed in this section ($n_y \geq n_u$ is enough for most expressions). The input DFT matrix $\mathbf{U}(\omega_k)$ is then given by

$$\mathbf{U}(\omega_k) = G_u(e^{j\omega_k T_s})(\mathbf{R}(\omega_k) - \mathbf{V}(\omega_k)), \quad (21)$$

where

$$G_u(q) = (I + F(q)G(q))^{-1}F(q). \quad (22)$$

A. Bias

Since $\mathbf{U}$ and $\mathbf{V}$ now are correlated, the FRF estimate $\hat{G}$ in (5) using the H_1 estimator will be biased. To analyze the bias, we will assume a closed loop setup according to (2) and (20), where the noise fulfills Assumption 1 and $\mathbf{R}$ and $\mathbf{V}$ are independent. The FRF estimation error is given by

$$\tilde{G} = \hat{G} - G = \mathbf{V}\mathbf{U}^\dagger = \mathbf{V}(\mathbf{R} - \mathbf{V})^\dagger G_u^{-1} \quad (23)$$

Calculating the bias $\mathbf{E} \tilde{G}$ seems hard due to the matrix inverse. Expanding $\mathbf{V}(\mathbf{R} - \mathbf{V})^\dagger$ gives

$$\begin{aligned} \tilde{G} &= \left[\frac{1}{n_e} \sum_{i=1}^{n_e} \mathbf{V}_i \mathbf{R}_i^H - \frac{1}{n_e} \sum_{i=1}^{n_e} \mathbf{V}_i \mathbf{V}_i^H \right] \times \\ &\quad \left[\frac{1}{n_e} \sum_{i=1}^{n_e} \mathbf{R}_i \mathbf{R}_i^H + \frac{1}{n_e} \sum_{i=1}^{n_e} \mathbf{V}_i \mathbf{V}_i^H \right. \\ &\quad \left. - \frac{1}{n_e} \sum_{i=1}^{n_e} \mathbf{R}_i \mathbf{V}_i^H - \frac{1}{n_e} \sum_{i=1}^{n_e} \mathbf{V}_i \mathbf{R}_i^H \right]^{-1} G_u^{-1} \end{aligned}$$

These sums of independent (over i) variables can, by the law of large numbers, be replaced by expectation as the number

of experiments, n_e , tends to infinity. Assuming that $\mathbf{R}$ and $\mathbf{V}$ are independent then gives

$$\lim_{n_e \rightarrow \infty} \tilde{G} = -\sigma_V^2 [\sigma_R^2 + \sigma_V^2]^{-1} G_u^{-1}, \quad (24)$$

where $\sigma_R^2 = \lim_{n_e \rightarrow \infty} \frac{1}{n_e} \sum_{i=1}^{n_e} \mathbf{R}_i \mathbf{R}_i^H$. Even though this asymptotic error expression is not technically the bias $E\tilde{G}$, it still gives valuable information about the bias one could expect when M is reasonably large. Using $G_u^{-1} = G + F^{-1}$ finally gives

$$\lim_{n_e \rightarrow \infty} \hat{G} = \sigma_R^2 [\sigma_R^2 + \sigma_V^2]^{-1} G - \sigma_V^2 [\sigma_R^2 + \sigma_V^2]^{-1} F^{-1}. \quad (25)$$

For frequencies where the SNR is poor, the estimate will tend to the inverse controller. If σ_V^2 and σ_R^2 are diagonal, (24) simplifies to

$$\lim_{n_e \rightarrow \infty} \tilde{G}_{ij} = -\frac{\sigma_{V,ii}^2}{\sigma_{V,ii}^2 + \sigma_{R,ii}^2} (G_{ij} + F_{ij}^{-1}), \quad (26)$$

such that the relative error ($\tilde{G}_{ij}/G_{ij}$) will be (approximately) proportional to the noise-to-signal ratio.

These bias expressions could be compared with the bias for the ARI estimator (11) in the SISO case [12]

$$E\tilde{G}^{\text{ARI}} = -e^{-\sigma_R^2/\sigma_V^2} G_u^{-1}, \quad (27)$$

and

$$E\hat{G}^{\text{ARI}} = (1 - e^{-\sigma_R^2/\sigma_V^2})G - e^{-\sigma_R^2/\sigma_V^2} F^{-1}, \quad (28)$$

which indicates a huge difference for large SNR's. A similar difference can be seen also in the MIMO case. When the SNR is small, on the other hand, the ARI estimator will deteriorate since $\mathbf{U}^{(m)}$ could lose rank for some blocks, giving infinite covariance. The covariance σ_G^2 for the ARI estimator (11) can easily be derived, similarly to the proof of Theorem 1,

$$\sigma_G^2 = \frac{1}{M^2} \sum_{m=1}^M [\mathbf{U}^{(m)} \mathbf{U}^{(m)H}]^{-T} \otimes \sigma_V^2, \quad (29)$$

which equals the covariance of the H_1 estimator (5) (cf. (16)) in case all $\mathbf{U}^{(m)}$ have the same amplitude. In other cases, the variance is typically larger for the ARI estimator. If the reference signal is measured, then the JIO estimator (12) is a good candidate since it is asymptotically ($M \rightarrow \infty$) unbiased and have approximately the same variance as the H_1 estimator. The selection of estimator is therefore, as usual, a bias-variance trade-off.

A problem with the asymptotic bias expressions (24) – (26) is that for large SNR's, n_e must be fairly large until these expressions are useful, mainly due to the variance of $X = \frac{1}{n_e} \sum_{i=1}^{n_e} \mathbf{V}_i (\mathbf{R}_i - \mathbf{V}_i)^H$. Assume, for simplicity, that σ_V^2 and σ_R^2 are diagonal. The matrix X , with mean $-\sigma_V^2$, then has variance

$$E X_{ij} X_{ij}^H = \frac{1}{n_e} (\sigma_{V,ii}^2 \sigma_{R,jj}^2 + \sigma_{V,ii}^2 \sigma_{V,jj}^2). \quad (30)$$

For this matrix to be diagonal dominant, we require

$$n_e \gg (\sigma_{R,jj}^2 + \sigma_{V,jj}^2) / \sigma_{V,ii}^2. \quad (31)$$

B. Covariance

For the remaining section, the bias will be neglected, using the approximation

$$\tilde{G} \approx \mathbf{V} \mathbf{R}^\dagger G_u^{-1}, \quad (32)$$

which can be viewed as neglecting the noise in the feedback loop as

$$\mathbf{U} \approx G_u \mathbf{R}. \quad (33)$$

Inserting (33) into Theorem 1 yields:

Theorem 2: Consider the H_1 estimator (5) and assume a closed loop setup according to (2) and (20), where the noise fulfills Assumption 1. For large SNR ($\mathbf{V} \ll \mathbf{R}$) the covariance $\sigma_G^2(\omega_k)$ is approximately given by

$$\sigma_G^2 = [G_u^{-H} (\mathbf{R} \mathbf{R}^H)^{-1} G_u^{-1}]^T \otimes \sigma_V^2, \quad (34)$$

with $G_u^{-1} = G + F^{-1}$.

Even though this covariance expression is approximate, it still gives valuable insight to how the covariance is affected by the closed loop setup. The validity of Theorem 2 will depend on the SNR. Now, similar expressions to Corollary 1 will be derived using the orthogonal multisine signal

$$\mathbf{R}(\omega_k) = [\mathbf{R}^{(1)}(\omega_k) \quad \dots \quad \mathbf{R}^{(M)}(\omega_k)], \quad (35a)$$

$$\mathbf{R}^{(m)}(\omega_k) = \mathbf{R}_{\text{diag}}^{(m)}(\omega_k) \mathbf{T}, \quad m = 1, \dots, M, \quad (35b)$$

as in (7) with $\mathbf{T}$ from (9). This parameterization corresponds to an optimal experiment design given output amplitude constraints, compared to the open loop case (7) when having input amplitude constraints. If we actually have input constraints in the closed loop case, $\mathbf{R} = G_u^{-1} \mathbf{U}$ can be used, yielding $\mathbf{R}_{\text{diag}} = G_u^{-1} \mathbf{U}_{\text{diag}}$. Note that $\mathbf{R}_{\text{diag}}$ then no longer is diagonal.

Corollary 2: Using an orthogonal multisine signal (35) with $\mathbf{T}$ from (9) simplifies σ_G^2 in Theorem 2 to

$$\sigma_G^2 = \left[G_u^{-H} \text{diag} \left\{ \frac{1}{n_u \sum_{m=1}^M |R_l^{(m)}|^2} \right\}_{l=1}^{n_u} G_u^{-1} \right]^T \otimes \sigma_V^2, \quad (36)$$

with $G_u^{-1} = G + F^{-1}$. The variance of each element in $\hat{G}$ can be written as

$$\sigma_{\hat{G}_{ij}}^2 = \sigma_{V,ii}^2 \sum_{l=1}^{n_u} \frac{|(G + F^{-1})_{lj}|^2}{n_u \sum_{m=1}^M |R_l^{(m)}|^2}. \quad (37)$$

One immediately notes that, to reduce the variance, the reference signal should be as large as possible. In addition, a large gain controller will also reduce the variance. Actually, $F = -G^{-1}$, would give zero error but is unrealistic since it corresponds to a marginally stable system and infinite input power. If the noise variance is the same for the different outputs, then all elements in a column will have the same variance. The variance will typically vary between different columns so for a symmetric system ($|G_{ij}| = |G_{ji}|$), the variance will usually be non-symmetric ($\sigma_{\hat{G}_{ij}}^2 \neq \sigma_{\hat{G}_{ji}}^2$). Small elements in a column will also, in general, have a larger relative error ($\sigma_{\hat{G}_{ij}}/|G_{ij}|$).

Fig. 1. Bias for the estimators with 22 dB SNR: H_1 (black line), ARI (gray line), JIO (thin black line), together with the system G (thick black line) and the bias expression (24) (thick gray line).

V. NUMERICAL ILLUSTRATION

As a numerical illustration, a simulation model of an industrial robot will be used. The model $G(q)$ is a linearized version of a nonlinear state-space model from [6] with 24 states, describing the dynamics of axes 1, 2 and 3 ($n_u = n_y = 3$) from applied motor torque to achieved motor velocity (the FRF of G can be seen in all the figures). The controller $F(q)$ is a diagonal PI controller with a gain of 8 dB in the excited frequency interval. A sample period of $T_s = 0.5$ ms is used with $N = N_P = 10^4$ samples from the steady state response.

As excitation, the orthogonal multisine signal (9) and (35) is used with a flat amplitude distribution from 2 to 60 Hz and $\mathbf{R}_{\text{diag}}^{(m)}$ the same in all $M = 100$ blocks. For the noise v in (2), $v_u = 0$ and v_y is normal distributed white noise, giving a diagonal covariance matrix σ_V^2 . The resulting signal-to-noise ratio, $\text{SNR}_i = 20 \log_{10}(\sigma_{R,ii}/\sigma_{V,ii})$, is constant for the excited frequencies and simulations with two different SNR's, 3 dB and 22 dB will be presented. The statistical properties of the estimators are obtained by repeating the simulations 100 times with different noise realizations.

First, consider the 22 dB case and the three estimators: H_1 (5), ARI (11) and JIO (12). The bias $E\tilde{G}$ and variance $\sigma_{\tilde{G}}^2$ (actually the standard deviation $\sigma_{\tilde{G}}$) of the estimators are plotted in Figs. 1 and 2, respectively. In addition are the expressions for bias (24) and variance (37) of the H_1 estimator plotted. As can be seen in Fig. 2, the variance is approximately the same for the three estimators and the approximate variance expression is fairly accurate.

As was mentioned in the end of Section IV, the variance is non-symmetric, which can be seen by comparing elements (1,3) and (3,1) in Fig. 2. For small elements, the relative error

Fig. 2. Standard deviation for the estimators with 22 dB SNR: H_1 (black line), ARI (gray line), JIO (thin black line), together with the system G (thick black line) and the variance expression (37) (thick gray line).

is also larger, which can be seen by comparing elements (1,1) and (3,1) in Fig. 2 for low frequencies. This is inherent in all the studied estimation methods.

For the bias expression, see Fig. 1, the bias of the ARI and JIO estimators are approximately the same and always equal to or lower than for the H_1 estimator. The asymptotic bias expression (24) is a lower limit for the bias of the H_1 estimator and there is a good match when G is large, especially for the diagonal elements. In this simulation, σ_V^2 and σ_R^2 are diagonal so the asymptotic expression (24) simplifies to (26). A problem is that with $M = 100$, the off-diagonal elements of $\mathbf{V}(\mathbf{R} - \mathbf{V})^\dagger$ in (23) are still quite large. If all elements of $\mathbf{V}(\mathbf{R} - \mathbf{V})^\dagger$ would be equal to α , then the bias is $\tilde{G}_{ij} = \alpha \sum_l (G_{lj} + F_{lj}^{-1})$, i.e., the same bias for all elements in a given column. That is almost the case here since $M = 100$ is not large enough, which explains the behavior for the off-diagonal elements in Fig. 1.

To see the different properties of the three estimators, the SNR is reduced to 3 dB and the bias and variance are plotted in Figs. 3 and 4, respectively. As can be seen, there is now a big difference both in bias and variance. The H_1 estimator gives largest bias and smallest variance and the JIO estimator gives smallest bias and a smaller variance than the ARI estimator. The JIO estimator should therefore be used if the reference signal is measured. The variance expression (37) predicts the variance of the JIO estimator fairly accurately, but the bias expression (24) does not work here. The validity of the bias expression with respect to the SNR therefore needs further studies.

Fig. 3. Bias for the estimators with 3 dB SNR: H_1 (black line), ARI (gray line), JIO (thin black line), together with the system G (thick black line) and the bias expression (24) (thick gray line).

VI. CONCLUSIONS

In this paper, FRF estimation methods for MIMO systems have been studied. The analysis has mainly focused on the H_1 estimator, where expressions for the asymptotic bias and approximate covariance for closed loop data have been proposed. In addition have the arithmetic mean estimator (ARI) and the joint input-output estimator (JIO) been generalized to the MIMO case. The JIO estimator requires the reference signal to be known, but is asymptotically unbiased.

In the simulations, all three estimators give approximately the same variance for large SNR's and the bias of the H_1 estimator is larger or equal to the bias of the ARI and JIO estimators. For small SNR's, the H_1 estimator gives largest bias and smallest variance and the JIO estimator gives smallest bias and a smaller variance than the ARI estimator. Among these three estimators, the JIO estimator is the preferred choice if the reference signal is measured. Otherwise, the ARI estimator should be used if the SNR is guaranteed to be large enough in all blocks (since each $U^{(m)}$ must be invertible, see also (29)). The H_1 estimator is, in that respect, more robust than the ARI estimator.

The proposed variance expression (37) predicts the variance fairly accurately, but the bias expression (24) does not work for small SNR's.

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Fig. 4. Standard deviation for the estimators with 3 dB SNR: H_1 (black line), ARI (gray line), JIO (thin black line), together with the system G (thick black line) and the variance expression (37) (thick gray line).

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Sammanfattning Abstract <div style="border: 1px solid black; padding: 10px; margin: 10px 0;"> <p>Estimation methods for the multivariable frequency response function are analyzed, both in open and closed loop. Approximate bias and covariance expressions are derived for the closed loop case. The usefulness of these expressions is illustrated in simulations of an industrial robot where three different estimators are compared. The choice of estimator depends on the signal-to-noise ratio as well as the measurement setup and a bias-variance trade-off.</p> </div>												
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