Computing the Controllability Function for Nonlinear Descriptor Systems

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Abstract

The computation of the controllability function for nonlinear descriptor systems is considered. Three different methods are derived. The first method is based on the necessary conditions for optimality from the Hamilton-Jacobi-Bellman theory for descriptor systems. The second method uses completion of squares to find the solution. The third method gives a series expansion solution, which with a finite number of terms can serve as an approximate solution.

Keywords: Optimal control, Algebraic/geometric methods, Constrained control, Descriptor Systems

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Abstract—The computation of the controllability function for nonlinear descriptor systems is considered. Three different methods are derived. The first method is based on the necessary conditions for optimality from the Hamilton-Jacobi-Bellman theory for descriptor systems. The second method uses completion of squares to find the solution. The third method gives a series expansion solution, which with a finite number of terms can serve as an approximate solution.

I. INTRODUCTION

During the last decades, descriptor systems have been extensively studied, see for example [1–3] and references therein. One reason is the natural formulation of many applications using this kind of system descriptions. The growing use of objected-oriented modeling languages such as MODELICA also increases the interest in these descriptions, since most often the output from such tools is in this form.

The topic of this paper is controllability functions for nonlinear descriptor systems in semi-explicit form

$$\dot{x}_1 = F_1(x_1, x_2, u) \quad (1a)$$

$$0 = F_2(x_1, x_2, u) \quad (1b)$$

where $x_1 \in \mathbb{R}^{n_1}$, $x_2 \in \mathbb{R}^{n_2}$, $u \in \mathbb{R}^m$, $F_1 : \mathbb{R}^{n_1+n_2+m} \mapsto \mathbb{R}^{n_1}$ and $F_2 : \mathbb{R}^{n_1+n_2+m} \mapsto \mathbb{R}^{n_2}$.

The controllability function is the minimum amount of control energy required to reach a specific state in infinite time. Hence, these functions measure how difficult a certain state is to obtain. The controllability function is defined as the solution to an optimal control problem. For state-space systems the background and theory can be found in [4] and references therein. A well-known fact for linear time-invariant state-space systems is that the controllability function is the same as the controllability gramian multiplied from the left and right by the state.

A phenomenon that may occur for descriptor systems but not for state-space systems, is that some combinations of x_1 and x_2 are not allowed due to certain inherent constraints. In [5] linear time-invariant descriptor systems are considered and a method to compute the controllability function is derived. There only combinations of x_1 and x_2 satisfying the constraints are investigated. We follow that line and use methods from the optimal control theory for descriptor systems to solve the optimal control problem corresponding to the controllability function.

Other concepts of controllability for descriptor systems have also been studied. A few such ideas can be found in, for example, [1, 6, 7].

Notation: The notation in this paper is fairly standard. In many cases x_1 and x_2 are grouped to one vector denoted

x , i.e., $x = (x_1^T, x_2^T)^T$. The Jacobian matrix $\frac{\partial V}{\partial x}$ is denoted V_x . $P \succ (\succeq) 0$ means that P is a real positive (semi)definite matrix.

II. CONTROLLABILITY FUNCTION

Basically a general controllability function should measure the amount of energy in the control signal $u(t)$ needed to reach a specific state x . Therefore, it is necessary to define a measure of the control signal energy. The most common energy measure, see for example [4], and the energy measure used throughout this paper is

$$J_c = \int_{-\infty}^0 m(u(t)) dt = \frac{1}{2} \int_{-\infty}^0 u(t)^T u(t) dt \quad (2)$$

However, it would be possible to use a more general $m(u(t))$, but in order to get a nice interpretation it has to satisfy $m(u(t)) > 0$ for all nonzero $u(t)$.

The controllability function $L_c(x_1)$ for the descriptor systems is defined as

$$L_c(x_1(0)) = \min_{u(\cdot)} J_c \quad (3)$$

subject to the system dynamics and

$$\begin{aligned} x_1(0) &= x_{1,0} \in \Omega \\ \lim_{t \rightarrow -\infty} x_1(t) &= 0 \end{aligned}$$

The function $L_c(x_{1,0})$ can be interpreted as the minimum amount of input energy required to drive the system from zero at $t = -\infty$ to $x_1(0) = x_{1,0}$ at $t = 0$.

The results in the paper will rely upon two assumptions used throughout the paper. Basically, the assumptions require the constraint equation (1b) to be solvable with respect to x_2 . The difference is the region for u on which the assumption must be satisfied. First we will use the following assumption.

Assumption 1: There is an open set $\Omega \subset \mathbb{R}^{n_1}$ containing the origin such that for all $x_1 \in \Omega$ and all u , (1b) can be solved to give

$$x_2 = \varphi(x_1, u), \quad x_1 \in \Omega, \quad u \in \mathbb{R}^m \quad (4)$$

Also $F_{2;x_2}(x_1, x_2, u)$ is nonsingular for all $x_1 \in \Omega$, x_2 and u solving (1b).

Hence, in this assumption F_2 must be solvable for all u . In the sections where a local controllability function is derived, a relaxed assumption will be used.

Assumption 2: It holds that $F_2(0, 0, 0) = 0$ and that $F_{2;x_2}(0, 0, 0)$ is nonsingular.

Together with the implicit function theorem, this assumption will guarantee that the constraint equation (1b) can be solved to give (4) locally around the origin.

It may seem like a serious limitation to only consider systems on semi-explicit form satisfying the assumptions above. However, in a series of papers, [8–10], it has been described how a rather general class of descriptor systems can be rewritten to satisfy the assumptions. Also the assumption of semi-explicitness is not that restrictive since many applications have this structure, see for example Example 5 in [10]. For a short discussion about the reduction procedure, see also the Appendix in [11].

As was mentioned in the introduction, we will only consider controllability within the set of consistent states. The set of consistent states is, based on the assumptions, the combinations of x_1 and x_2 which satisfy the constraints for some u , i.e.,

$$\mathcal{N} = \{(x_1, x_2) \mid x_1 \in \Omega, x_2 = \varphi(x_1, u), u \in \Omega_u \subset \mathbb{R}^m\}$$

Therefore, the final state $x_{1,0}$ is chosen in Ω . Then, from the assumptions it is known that there exist a x_2 and an u satisfying (1b).

Remark 1: If it is possible to reach all $x_{1,0} \in \Omega$, it is also possible to reach all $(x_{1,0}, x_{2,0}) \in \mathcal{N}$. The reason is that it is possible to use some $u(t)$ for $-\infty < t < 0$ and then at $t = 0$ let $u(0)$ be the value such that $x_2(0) = x_{2,0}$.

One further assumption will be made throughout the paper.

Assumption 3: The system (1) has an equilibrium at $x_1 = 0$, $x_2 = 0$, $u = 0$, i.e., $F_1(0, 0, 0) = 0$ and $F_2(0, 0, 0) = 0$. Notice that this assumption does not introduce any loss of generalization. It is always possible, using a state transformation, to move the stationary point to the origin.

As earlier mentioned the problem of finding the feedback law $u(x_1)$ minimizing (3) subject to the given constraints is an optimal control problem. In this work, three different methods to find $L_c(x_1)$ will be derived. The first method is based on the necessary conditions from the Hamilton-Jacobi-Bellman theory. This method yields a solution for $x_1 \in \Omega$. The second method is based on an approach similar to the one used in [4]. It uses completion of squares and is possible to use due to the fact that the performance criterion (3) only includes the term $u^T u$. This approach also yields a solution holding for $x_1 \in \Omega$. Thirdly, a method is derived that finds a local solution of $L_c(x_1)$, i.e., a solution that holds in a neighborhood of the origin. The last method can also be used to find an approximate solution for the controllability function.

III. REVIEW OF OPTIMAL FEEDBACK CONTROL FOR DESCRIPTOR SYSTEMS

An optimal control problem for a descriptor system is defined by a performance criterion, the dynamics and some boundary conditions. The performance criterion used in this paper has the form

$$J = \int_0^\infty L(x_1, x_2, u) dt \quad (5)$$

and hence an infinite time horizon is assumed. The optimal control problem is formulated as

$$V(x_1(0)) = \min_{u(\cdot)} J \quad (6)$$

subject to the dynamics (1) and the boundary conditions

$$\begin{aligned} x_1(0) &= x_{1,0} \in \Omega \\ \lim_{t \rightarrow \infty} x_1(t) &= 0 \end{aligned}$$

A control law u expressed as a feedback law from x_1 and x_2 may change the invertibility of F_2 with respect to x_2 for the closed loop system. However, with our assumptions it is known that x_2 can be solved for even for the closed loop system. Then, x_1 will be the free variables and x_2 is chosen consistently, i.e., such that $F_2(x_1, x_2, u(x_1, x_2)) = 0$.

We will in this work only consider feedback laws such that $x_1(\infty) = 0$, i.e., the closed loop system is required to be asymptotically stable. To verify this, several methods can be used. One method, based on the implicit function theorem, is described in [12]. Another method, which applies for polynomial systems can be found in [13]. One further method, based on Lyapunov-like equations, is described in [14].

This optimal control problem has been studied in [11] and there it is proven that if the system satisfies Assumption 1 the necessary conditions become

$$0 = L_u + W_1 F_{1;u} + W_2 F_{2;u} \quad (7a)$$

$$0 = L + W_1 F_1 \quad (7b)$$

$$0 = F_2 \quad (7c)$$

$$0 = W_2 + (L_{x_2} + W_1 F_{1;x_2}) F_{2;x_2}^{-1} \quad (7d)$$

where $W_1(x_1)$ is the gradient of some continuously differentiable function $V(x_1)$, $W_2(x_1, x_2)$ is a continuous function. The rest of the functions in the right hand sides are evaluated at x_1 , x_2 and u .

IV. METHOD BASED ON THE NECESSARY CONDITIONS FROM THE HAMILTON-JACOBI-BELLMAN THEORY

Consider the system (1) and the performance criterion (3). This is a special case of the optimal control problem defined in Section III, with

$$L(x_1, x_2, u) = \frac{1}{2} u^T u \quad (8)$$

However, since the final state and not the initial state is specified, the time in (3) can be considered as running backwards compared to (6). This fact yields that some signs are changed in the necessary conditions (7). Further, because the cost function (8) has the given structure the necessary conditions can be simplified. The result is formulated as a proposition.

Proposition 1: Assume that the system (1) satisfies Assumption 1 and 3. The necessary conditions for the controllability function can then be written as

$$0 = u^T - W_1 F_{1;u} - W_2 F_{2;u} \quad (9a)$$

$$0 = \frac{1}{2} u^T u - W_1 F_1 \quad (9b)$$

$$0 = F_2 \quad (9c)$$

$$0 = W_2 + W_1 F_{1;x_2} F_{2;x_2}^{-1} \quad (9d)$$

where $W_1(x_1)$ is the gradient of some continuously differentiable function $V(x_1)$, $W_2(x_1, x_2)$ is a continuous function. The rest of the functions are evaluated at x_1 , x_2 and u .

Remark 2: A special case where the equations in Proposition 1 become extra simple is

$$\dot{x}_1 = f_1(x_1, x_2) + g_1(x_1)u \quad (10a)$$

$$0 = f_2(x_1, x_2) + g_2(x_1)u \quad (10b)$$

First notice that $f_{2;x_2}(x_1, x_2)$ is nonsingular for all (x_1, x_2) such that $f_2(x_1, x_2) = 0$ is solvable since $F_{2;x_2}(x_1, x_2, u)$ is nonsingular for all $(x_1, x_2, u) \in \Omega$ and then particularly for $u = 0$.

Using (9d) an expression for $W_2(x_1, x_2)$ can be formulated as

$$W_2(x_1, x_2) = -W_1(x_1)f_{1;x_2}(x_1, x_2)f_{2;x_2}^{-1}(x_1, x_2)$$

Combining this expression with (9a) yields

$$u = \hat{g}(x_1, x_2)^T W_1(x_1)^T$$

and after some more manipulations the necessary conditions can be rewritten as

$$0 = W_1(x_1)\hat{f}(x_1, x_2) + \frac{1}{2}W_1(x_1)\hat{g}(x_1, x_2)\hat{g}(x_1, x_2)^T W_1(x_1)^T \quad (11a)$$

$$0 = f_2(x_1, x_2) + g_2(x_1)\hat{g}(x_1, x_2)^T W_1(x_1)^T \quad (11b)$$

where

$$\begin{aligned} \hat{f}(x_1, x_2) &= f_1(x_1, x_2) - f_{1;x_2}(x_1, x_2)f_{2;x_2}^{-1}(x_1, x_2)f_2(x_1, x_2) \\ \hat{g}(x_1, x_2) &= g_1(x_1) - f_{1;x_2}(x_1, x_2)f_{2;x_2}^{-1}(x_1, x_2)g_2(x_1) \end{aligned}$$

Hence, the original four equations with four unknowns are reduced to the two equations (11) and the two unknowns $W_1(x_1)$ and $x_2 = \eta(x_1)$.

V. METHOD BASED ON COMPLETION OF SQUARES

In Section IV necessary conditions for the controllability function were found. A solution to these conditions must then be verified using some method to prove that it actually is the controllability function. In this section one method yielding sufficient conditions will be derived for a special class of descriptor system. The considered class are supposed to have the form

$$E\dot{x} = f(x) + g(x)u \quad (12)$$

where $E = \begin{pmatrix} I_{n_1} & 0 \\ 0 & 0 \end{pmatrix}$. The underlying property which makes it possible to derive the sufficient conditions, is that the performance criterion only depends on the squared control signal, i.e., $u^T u$. The result is stated as a theorem.

Theorem 2: Suppose there exist continuous functions $W_1(x_1) = V_{x_1}(x_1)$ and $W_2(x_1, x_2)$ such that $\tilde{L}_c(x_1) = (W_1(x_1) \quad W_2(x_1, x_2))$ fulfills

$$0 = \tilde{L}_c(x)f(x) + \frac{1}{2}\tilde{L}_c(x)g(x)g(x)^T \tilde{L}_c(x)^T \quad (13)$$

for all $x \in \mathcal{N}$. Furthermore, assume that for the control choice

$$u = g(x)^T \tilde{L}_c(x)^T \quad (14)$$

the system (12) can be solved backwards in time from $t = 0$, with $x(t) \rightarrow 0, t \rightarrow -\infty$. Then, $L_c(x_1) = V(x_1)$ and the corresponding u is the optimal control law.

Proof: Assume that $x_{1,0} \in \Omega$. For any control signal u such that the solution to (12) fulfills $x(t) \rightarrow 0$ as $t \rightarrow -\infty$ it follows

$$\begin{aligned} \frac{1}{2} \int_{-\infty}^0 u^T u dt &= V(x_1(0)) \\ &+ \int_{-\infty}^0 \left(\frac{1}{2} u^T u - V_{x_1}(f_1 + g_1 u) - W_2(f_2 + g_2 u) \right) dt \end{aligned}$$

where $V(x_1)$, $W_2(x_1, x_2)$ are arbitrary sufficiently smooth functions. Completing the squares gives

$$\frac{1}{2} \int_{-\infty}^0 u^T u dt = V(x_1(0)) + \int_{-\infty}^0 \frac{1}{2} \|u - g(x)^T \tilde{L}_c(x)^T\|^2 dt$$

provided (13) is satisfied. As described in [4], $V(x_1(0))$ is a lower bound for the integral in (3). By choosing $u = g(x)^T \tilde{L}_c(x)^T$ this lower bound is obtained and since this control choice is such that the closed loop system can be solved backwards in time and $x(-\infty) = 0$, it is optimal. Therefore, for all $x_{1,0} \in \Omega$, ■

$$L_c(x_1(0)) = \min_{u(\cdot)} \frac{1}{2} \int_{-\infty}^0 u^T u dt = V(x_1(0))$$

One requirement was that the closed loop system, using (14), is asymptotically stable in backwards time around the origin for $x \in \mathcal{N}$. This is equivalent to

$$E\dot{\tilde{x}} = -(f(\tilde{x}) + g(\tilde{x})g(\tilde{x})^T \tilde{L}_c(\tilde{x})^T) \quad (15)$$

being uniquely solvable and asymptotically stable, where $\tilde{x}(s) = x(-t)$. To verify that (15) is asymptotically stable the methods described in Section III can be used.

Remark 3: If the system has the form in (12) it is often a good idea to combine the method in this section with the method in Section IV. First candidate solutions are found using the necessary conditions and then the optimal solution is chosen by Theorem 2.

VI. REVIEW OF LOCAL SOLUTIONS TO THE OPTIMAL CONTROL PROBLEM

In many cases, the necessary conditions in (7) or Proposition 1 can be very hard to solve globally. Then, it may be interesting to look for a local solution, i.e., a solution valid only in a neighborhood around some point. In [15] the optimal control problem (6) is considered and some conditions for such a solution to exist is derived. Since the controllability function is a special case of an optimal control problem, a local solution should in principle be possible to compute using the method described in that paper. A problem is that the cost matrix in the controllability function does not satisfy the assumptions made in [15]. However, in this paper it will be shown that by modifying the proof it is possible to show similar results. First, a short resume of the idea and the results in [15] is given.

We will first make an assumption.

Assumption 4: The functions $F(x, u)$ and $L(x, u)$ in (6) are analytic functions in some neighborhood of the origin, $x = 0, u = 0$.

This assumption guarantees that the functions can be expanded in convergent power series

$$F(x, u) = Ax + Bu + F_h(x, u) \quad (16a)$$

$$L(x, u) = \frac{1}{2}x^T Qx + x^T Su + \frac{1}{2}u^T Ru + L_h(x, u) \quad (16b)$$

where the matrices A, B, Q, S are partitioned as

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix},$$

$$Q = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix}, \quad S = \begin{pmatrix} S_1 \\ S_2 \end{pmatrix}$$

and $F_h(x, u)$ and $L_h(x, u)$ contain higher order terms of at least degree two and three respectively. Using Assumption 2 it is known that locally we have

$$x_2 = \varphi(x_1, u) = -A_{22}^{-1}A_{21}x_1 - A_{22}^{-1}B_2u + \varphi_h(x_1, u) \quad (17)$$

where $\varphi_h(x_1, u)$ contains terms of degree two or higher. The first order term in (17) will locally define a change of variables around the origin given by

$$\begin{pmatrix} x_1 \\ x_2 \\ u \end{pmatrix} = \Pi \begin{pmatrix} x_1 \\ u \end{pmatrix} = \begin{pmatrix} I & 0 \\ -A_{22}^{-1}A_{21} & -A_{22}^{-1}B_2 \\ 0 & I \end{pmatrix} \begin{pmatrix} x_1 \\ u \end{pmatrix}$$

If (17) is applied to (16) the result is

$$\dot{x}_1 = \hat{A}x_1 + \hat{B}u + \hat{f}_{1h}(x_1, u) \quad (18a)$$

$$\hat{L}(x_1, u) = \frac{1}{2} \begin{pmatrix} x_1 \\ u \end{pmatrix}^T \begin{pmatrix} \hat{Q} & \hat{S} \\ \hat{S}^T & \hat{R} \end{pmatrix} \begin{pmatrix} x_1 \\ u \end{pmatrix} + \hat{L}_h(x_1, u) \quad (18b)$$

where

$$\hat{A} = A_{11} - A_{12}A_{22}^{-1}A_{21} \quad \hat{B} = B_1 - A_{12}A_{22}^{-1}B_2 \quad (19a)$$

$$\begin{pmatrix} \hat{Q} & \hat{S} \\ \hat{S}^T & \hat{R} \end{pmatrix} = \Pi^T \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} \Pi \quad (19b)$$

and the higher order terms $\hat{f}_{1h}(x_1, u)$ and $\hat{L}_h(x_1, u)$ can be found in [15]. The conditions under which a local solution exists can be formulated as a theorem.

Theorem 3: Consider the optimal control problem (6). Assume that the cost matrix satisfies $\begin{pmatrix} \hat{Q} & \hat{S} \\ \hat{S}^T & \hat{R} \end{pmatrix} \succ 0$. Then the following statements are equivalent:

- 1) $(\hat{A}, \hat{B})$ is stabilizable.
- 2) The optimal control problem (6) has a local solution in a neighborhood of the origin.

Proof: See [15]. ■

The local optimal solution is given by series expansions as

$$V(x_1) = \frac{1}{2}x_1^T Px_1 + V_h(x_1) \quad (20a)$$

$$u(x_1) = Dx_1 + u_h(x_1) \quad (20b)$$

where P is the unique positive definite solution to (21a) such that D in (21b) makes the real parts of the eigenvalues of $\hat{A} + \hat{B}D$ negative.

$$P\hat{\tilde{A}} + \hat{\tilde{A}}^T P - P\hat{B}\hat{R}^{-1}\hat{B}^T P + \hat{Q} - \hat{S}\hat{R}^{-1}\hat{S}^T = 0 \quad (21a)$$

$$D + \hat{R}^{-1}(\hat{S}^T + \hat{B}^T P) = 0 \quad (21b)$$

From [15] it is given that $\hat{\tilde{A}} = \hat{A} - \hat{B}\hat{R}^{-1}\hat{S}^T$ in (21). The expressions for the higher order terms $V_h(x_1)$ and $u_h(x_1)$ can be found in [15].

VII. METHOD TO FIND A LOCAL SOLUTION FOR THE CONTROLLABILITY FUNCTION

In Theorem 3, the cost matrix must be positive definite. This fact makes it necessary to modify the theorem before it applies to the problem of finding a controllability function. As will be seen in this section, by using some additional assumptions it is still possible to guarantee the existence of a local solution. The result is formulated as a theorem.

Theorem 4: Assume that the system satisfies Assumption 4, that all eigenvalues of $\hat{A}$ have negative real parts and that $(\hat{A}, \hat{B})$ is controllable. Then the system has a local controllability function given by

$$L_c(x_1) = \frac{1}{2}x_1^T G_c x_1 + L_{ch}(x_1)$$

where G_c is the unique positive definite solution to the algebraic Riccati equation (ARE)

$$0 = \hat{A}^T G_c + G_c \hat{A} + G_c \hat{B} \hat{B}^T G_c \quad (22a)$$

where $\hat{A}$ and $\hat{B}$ are given in (19a). Expressions which can be used to compute the higher order terms of $L_{ch}(x_1)$ can be found in [16]. The unique positive definite solution G_c is also such that using

$$D = \hat{B}^T G_c x_1 \quad (22b)$$

in the feedback control (20b), the closed loop system

$$\dot{x}_1 = -(\hat{A} + \hat{B} \hat{B}^T G_c)x_1 + f_{clh}(x_1)$$

is locally asymptotically stable.

Proof: The results in [15] are based on the proof in [17]. Careful examination of the proof in [17] shows that most parts of the proof still hold when x is not present in L , but two parts remain to be proven.

In the controllability problem we have that $Q = 0$, $S = 0$ and $R = I$ and that the time is going backwards compared to the optimal control problem (6). Therefore, when using the results in Section III or in [17], the system will be

$$\dot{x}_1 = -\hat{A}x_1 - \hat{B}u - \hat{f}_{1h}(x_1) \quad (23a)$$

and the cost matrix and $\tilde{\tilde{A}}$ are

$$\tilde{\tilde{A}} = -\hat{A}, \quad \begin{pmatrix} \hat{Q} & \hat{S} \\ \hat{S}^T & \hat{R} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \quad (23b)$$

Further, it can be noted that for a feedback law to be optimal in the controllability problem, it is necessary that (18a) is made locally asymptotically stable, i.e., $\Re \lambda(-\hat{A} - \hat{B}D) < 0$. Otherwise, the cost function J_c cannot converge locally around the origin.

The first part to be shown is that P in the expression for the cost function found in [17]

$$J_c(x_{1,0}) = \frac{1}{2}x_{1,0}^T P x_{1,0} + j_h(x_{1,0})$$

still is at least positive semidefinite. Using only feedback laws satisfying the requirements above and inserting the data in (23) it follows that

$$P = \int_0^\infty e^{(-\hat{A} - \hat{B}D)^T t} D^T D e^{(-\hat{A} - \hat{B}D)t} dt$$

where P is obviously positive semidefinite since it is the integral of an expression which is positive semidefinite. In fact it can even be shown that P is positive definite under the given assumptions. However, this will not be necessary for the proof.

The first terms in the local solution of $L_c(x_1)$ are given by the same expressions as in Theorem 3, *i.e.*, (20b). However, using the data in (23), the ARE (21a) is changed to the ARE (22a). From the first part of the proof it is known that the optimal solution must fulfill $P \succeq 0$ and that the corresponding D from (21b) has to be such that $\Re\lambda(-\hat{A} - \hat{B}D) < 0$. Hence, the second part necessary to prove is that given the assumptions above there exists such a solution.

The ARE in (22a) is somewhat special since there is no constant term. From the assumptions, two properties of the system are known. Firstly, it is certain that $(-\hat{A}, -\hat{B})$ is stabilizable, since it is assumed that $(\hat{A}, \hat{B})$ is controllable. Secondly, it is known that $(-\hat{A}, 0)$ has no undetectable modes on the imaginary axis, because of asymptotic stability of $\hat{A}$. These two properties will, according to [18], yield that (22a) has a unique maximal positive definite solution, G_c . Furthermore, this solution is such that $-\hat{A} - \hat{B}\hat{B}^T G_c$ is asymptotically stable.

The stabilizing solution, G_c , must also be the only positive definite solution. To realize this, we first note that for all $G_c \succ 0$, (22a) can be reformulated as a Lyapunov equation

$$0 = \hat{A}G_c^{-1} + G_c^{-1}\hat{A}^T + \hat{B}\hat{B}^T$$

Hence, all positive definite solutions to the ARE must also be a positive definite solution to the Lyapunov equation and vice versa. However, it is well-known that the Lyapunov equation has a unique positive definite solution if $\hat{A}$ is asymptotically stable and $(\hat{A}, \hat{B})$ is controllable, see for example [19].

Furthermore, according to [18], all other positive semidefinite solutions to (27a) are such that $-\hat{A} - \hat{B}\hat{B}^T G_c$ will have some eigenvalues with positive real part.

Therefore, since G_c had to be at least positive semidefinite from the first part of the proof, it means that by choosing the unique positive definite solution the controllability function is found. ■

Remark 4: The local solution satisfies $L_c(x_1) > 0$ for all nonzero x_1 in some neighborhood of the origin.

Remark 5: By using only a finite number of terms in the series solution of $L_c(x)$ an approximate solution is found.

Remark 6: In [16], the results above are extended to handle also systems not given in semi-explicit form.

VIII. LINEAR DESCRIPTOR SYSTEMS

In this section the methods described in Section II are applied to linear descriptor systems. Another approach would be to consider the linear case as a special case of the result in Section VII, but the objective with this section is to show the ideas with the methods in Section II.

It should be pointed out that the purpose is also only to show the ideas since the theory, and a method to compute the controllability function, already are presented in [5].

Suppose we have a linear descriptor system

$$\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} u \quad (24)$$

satisfying Assumption 1, *i.e.*, A_{22} is nonsingular. Then

$$x_2 = \varphi(x_1, u) = -A_{22}^{-1}A_{21}x_1 - A_{22}^{-1}B_2u$$

for all $x_1 \in \mathbb{R}^{n_1}$ and all $u \in \mathbb{R}^m$. Hence, the reduced system becomes

$$\dot{x}_1 = \hat{A}x_1 + \hat{B}u \quad (25)$$

where $\hat{A}$ and $\hat{B}$ are given in (19a). It will be assumed that (25) is controllable and that (25) is asymptotically stable with $u(t) \equiv 0$.

Remark 7: Controllability of (25) is equivalent to so-called R-controllability of (24). Furthermore, asymptotic stability of (25) with $u(t) \equiv 0$ is equivalent to that (24) is asymptotically stable in descriptor sense using $u(t) \equiv 0$, see [1].

In order to compute the controllability function for a linear index one descriptor system the method described in Section IV is applied. The optimal feedback has to fulfill the set of equations

$$0 = u^T - V_{x_1}(x_1)B_1 - W_2(x_1, x_2)B_2 \quad (26a)$$

$$0 = 1/2u^T u - V_{x_1}(x_1)(A_{11}x_1 + A_{12}x_2 + B_1u) \quad (26b)$$

$$0 = A_{21}x_1 + A_{22}x_2 + B_2u \quad (26c)$$

$$0 = V_{x_1}(x_1)A_{12} + W_2(x_1, x_2)A_{22} \quad (26d)$$

After some manipulation and if we assume that $V(x_1) = \frac{1}{2}x_1^T G_c x_1$, it can be shown that (26) has a solution for all x_1 , *i.e.*, $\Omega = \mathbb{R}^{n_1}$, if and only if

$$0 = \hat{A}^T G_c + G_c \hat{A} + G_c \hat{B}\hat{B}^T G_c \quad (27a)$$

has a solution. The corresponding feedback law is given by

$$u = \hat{B}^T V_{x_1} = \hat{B}^T G_c x_1 \quad (27b)$$

and $W_2(x_1)$ is found as

$$W_2(x_1) = x_1^T G_c A_{12} A_{22}^{-1}$$

Above only necessary conditions are considered. However, if the feedback law (27b) is such that

$$\dot{x}_1 = -(\hat{A} + \hat{B}\hat{B}^T G_c)x_1 \quad (28)$$

is asymptotically stable it is possible, for example using Theorem 2, to show that the optimal feedback law is found.

The ARE (22a) is the same as in Section VII and the assumptions are also the same. Therefore, it is known that there exists a unique positive definite solution G_c such that the closed loop system (28) is asymptotically stable and it follows that

$$L_c(x_1) = V(x_1) = \frac{1}{2}x_1^T G_c x_1$$

IX. EXAMPLES

In order to illustrate the method of computing the controllability function, we will study an example.

Fig. 1. The rolling disc.

The system dynamics is given by the set of differential and algebraic equations

$$\dot{z}_1 = z_2 \quad (29a)$$

$$\dot{z}_2 = -\frac{k_1}{m}z_1 - \frac{k_2}{m}z_1^3 - \frac{b}{m}z_2 + \frac{1}{m}\lambda \quad (29b)$$

$$\dot{z}_3 = -\frac{r}{J}\lambda + \frac{1}{J}u \quad (29c)$$

$$0 = z_2 - rz_3 \quad (29d)$$

and describes a rolling disc, that rolls on a surface without slipping, see Figure 1. The disc is connected to a fixed wall with a nonlinear spring and a linear damper. The spring has the coefficients k_1 and k_2 , which both are positive. The damping coefficient of the damper is given by b and is also positive. The radius of the disc is r , its inertia is given by J and the mass of the disc is m . This system description does not satisfy Assumption 1 or 2. However, using the method in [10], (29) can be rewritten as the following description

$$\dot{z}_1 = z_2 \quad (30a)$$

$$\dot{z}_2 = -\frac{k_1}{m}z_1 - \frac{k_2}{m}z_1^3 - \frac{b}{m}z_2 + \frac{1}{m}\lambda \quad (30b)$$

$$0 = z_2 - rz_3 \quad (30c)$$

$$0 = -\frac{k_2}{m}z_1^3 - \frac{k_1}{m}z_1 - \frac{b}{m}z_2 + \left(\frac{r^2}{J} + \frac{1}{m}\right)\lambda + \frac{-r}{J}u \quad (30d)$$

which satisfies the assumptions. We will use the following abbreviation $x = [z_1 \ z_2 \ z_3 \ \lambda]^T$ and by comparing with (1) it can be seen that $x_1 = [z_1 \ z_2]^T$ and $x_2 = [z_3 \ \lambda]^T$.

This system has a form suitable for the result in Section V. Therefore, we will first use Proposition 1 in Section IV to find possible solutions and then Theorem 2 in Section V to choose which solution that is optimal.

From (9d) we have that W_2 must satisfy

$$W_2(x) = -V_{x_1}(x_1)F_{1;x_2}F_{2;x_2}^{-1} = -V_{x_1}\begin{pmatrix} 0 & 0 \\ 0 & \frac{J}{J+mr^2} \end{pmatrix} \quad (31)$$

Since F_1 does not depend on u , (9a) becomes

$$u = F_{2;u}(x, u)^T W_2(x)^T = \begin{pmatrix} 0 & \frac{r}{J+mr^2} \end{pmatrix} V_{x_1}(x_1)^T \quad (32)$$

For (30) it is possible to compute $x_2 = \varphi(x_1, u)$ explicitly, using the last two rows in (30), as

$$z_3 = \frac{1}{r}z_2 \quad (33)$$

$$\lambda = \left(\frac{r^2}{J} + \frac{1}{m}\right)^{-1} \left(\frac{k_1}{m}z_1 + \frac{k_2}{m}z_1^3 + \frac{b}{m}z_2 + \frac{r}{J}u\right)$$

We combine (33) and (32) into (9b) and assign $V(x_1) = a_1 z_1^2 + a_2 z_1^4 + a_3 z_2^2$. Solving (9b) then give

$$V(x_1) = bk_1 r^2 z_1^2 + \frac{1}{2}bk_2 r^2 z_1^4 + b(J + mr^2)z_2^2 \quad (34)$$

or the trivial solution $V(x_1) = 0$. Back-substitution of (34) into (31) and (32) yields

$$W_2(x) = \begin{pmatrix} 0 & -2bJz_2 \end{pmatrix}, \quad u(x_1) = 2brz_2 \quad (35)$$

The system is polynomial, and for given values of the parameters it would be possible to use the method in [13] to show asymptotic anti-stability of (30) with the control choice (35). However, here we choose to show stability using closed loop reduced system in backward time

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} -\frac{k_1}{\frac{J}{r^2}+m}z_1 + \frac{k_2}{\frac{J}{r^2}+m}z_1^3 + \frac{b}{\frac{J}{r^2}+m}z_2 - \frac{1}{(\frac{J}{r^2}+m)r}2brz_2 \end{pmatrix}$$

where the right hand side will be denoted $F_{red,cl}$ in the sequel. For this system, $V(x_1)$ is a Lyapunov function since

$$V(x_1) = bk_1 r^2 z_1^2 + \frac{1}{2}bk_2 r^2 z_1^4 + b(J + mr^2)z_2^2 > 0$$

$$-V_{x_1}(x_1)F_{red,cl}(x_1) = -2b^2 r^2 z_2^2 < 0$$

for all $x_1 \neq 0$, since if the $V_{x_1}(x_1)F_{red,cl}(x_1) = 0$ it requires that $z_2 = 0$, but then $z_1 = 0$ due to that k_1 and k_2 are positive. Therefore, Theorem 2 is fulfilled and it is certain that

$$L_c(x_1) = V(x_1)$$

with $u(x_1)$ chosen as (35).

In this case, the exact controllability function is polynomial. Therefore, the same solution would be obtained using the series expansion method. More examples showing the different methods can be found in [16].

X. CONCLUSIONS

In this paper, the controllability function for nonlinear descriptor systems is considered. Three different methods to compute the controllability function are derived. The first two methods yield explicit expressions for the solution $L_c(x_1)$, while the third method yields a series expansion solution. The third method can also be used to find an approximate solution to $L_c(x_1)$, by truncating the series expansion.

A limitation is that only systems satisfying either Assumption 1 or 2 in semi-explicit form are handled. However, by using index reduction methods more general systems can be handled as shown in the example.

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Sammanfattning Abstract The computation of the controllability function for nonlinear descriptor systems is considered. Three different methods are derived. The first method is based on the necessary conditions for optimality from the Hamilton-Jacobi-Bellman theory for descriptor systems. The second method uses completion of squares to find the solution. The third method gives a series expansion solution, which with a finite number of terms can serve as an approximate solution.					
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