Solution for the “Modeling and Simulation" exam (TSRT62) 2020-01-10

1. (a) Two reasons. First, the heuristic rule TSTF “Try simple things first” should always be applied (hence ARX being simpler than ARMAX and BJ is the first candidate). Second ARX parameters can be found quickly and cheaply by the least-squares method (not true for ARMAX or BJ).

   (b) Applying the scaling $z = \alpha x$ and $t = \beta\tau$, one gets

$$\frac{dz}{d\tau} = -\frac{2}{\alpha^2\beta}z^3 + \frac{\alpha}{\beta}5$$

   which leads to $\alpha = (2/5)^{1/3}$ and $\beta = 2^{1/3}5^{2/3}$.

   (c) You need 117 state variables, for instance $x_1 = y, x_2 = \dot{y}, \ldots, x_{117} = y^{(116)}$.

   (d) Let us study the test equation $\dot{x} = \lambda x$. For the backward Euler method $x_{n+1} = x_n + hf(x_{n+1})$, it is $x_{n+1} = (1-h\lambda)^{-1}x_n$. The stability region is given by $|1-h\lambda| > 1$, i.e., the region outside a circle of radius 1 centered at 1. Since the system has all eigenvalues on the negative half-plane $(-1, -10, -100)$, the numerical solution will be stable for all positive h.

   (e) Replace the fastest mode (state x_2 of eigenvalue -100) with the static expression $0 = -100x_2 + 10u$. This gives $x_2 = 0.1u$, which inserted in the original model gives the approximate description

$$\begin{aligned} \dot{x}_1 &= -10x_1 + 8u \\ y &= 10x_1 + 10u \end{aligned}$$

2. (a)

$$\begin{aligned} R_y(1) &= E(y(t)y(t-1)) = E\Big((0.5y(t-1) + u(t-1) + e(t))y(t-1)\Big) \\ &= 0.5R_y(0) + R_{yu}(0) + R_{ye}(-1) \end{aligned}$$

   Since $y(t)$ does not depend on $u(t)$ (but only on past input values $u(t-1), u(t-2), \ldots$), $R_{yu}(0) = 0$. Similarly, $R_{ye}(\tau) = 0$, $\tau < 0$. Hence

$$R_y(1) = 0.5R_y(0)$$

   (b) Using the prediction error minimization approach, the asymptotic estimate of the parameter vector

$$\theta = \begin{bmatrix} a_1 \\ a_2 \\ b \end{bmatrix}$$

is given by

$$\hat{\theta} = \lim_{N\to\infty} \text{argmin}_\theta \frac{1}{N} V_N(\theta) = \lim_{N\to\infty} \text{argmin}_\theta \frac{1}{N} \sum_{t=1}^{N} (y(t) - \hat{y}(t|\theta))^2$$

where

$$\begin{aligned} y(t) - \hat{y}(t|\theta) &= 0.5y(t-1) + u(t-1) + e(t) - a_1 y(t-1) - a_2 y(t-2) - bu(t-1) \\ &= (0.5 - a_1)y(t-1) - a_2 y(t-2) + (1-b)u(t-1) + e(t) \end{aligned}$$

When $N \to \infty$, $V_N(\theta) \to W(\theta) = E(y(t) - \hat{y}(t|\theta))^2$. Computing explicitly

$$\begin{aligned} E(y(t) - \hat{y}(t|\theta))^2 =& (0.5 - a_1)^2 R_y(0) + a_2^2 R_y(0) + (1-b)^2 R_u(0) + R_e(0) \\ &+ 2(0.5 - a_1)a_2 R_y(1) + 2(1-b)(0.5-a_1)R_{yu}(0) - 2(1-b)a_2 R_{yu}(-1) \\ &+ 2(0.5-a_1)R_{ye}(-1) - 2a_2 R_{ye}(-2) + 2(1-b)R_{ue}(-1) \end{aligned}$$

Since e and u are independent, $R_{eu}(\tau) = 0$. Since $y(t)$ and $y(t-1)$ do not depend on $u(t)$, $R_{yu}(0) = R_{yu}(-1) = 0$. Similarly, $R_{ye}(\tau) = 0$, $\tau < 0$. Hence

$$\lim_{N\to\infty} V_N(\theta) = ((0.5 - a_1)^2 + a_2^2)R_y(0) + (1-b)^2 \lambda_u + \lambda_e + 2(0.5 - a_1)a_2 R_y(1)$$

Using $R_y(1) = 0.5R_y(0)$,

$$W(\theta) = \lim_{N\to\infty} V_N(\theta) = \Big((0.5 - a_1)^2 + a_2^2 + (0.5 - a_1)a_2\Big) R_y(0) + (1-b)^2 \lambda_u + \lambda_e$$

Since $R_y(0)$ is a variance, obviously $R_y(0) \geq 0$. To see what values of a_1, a_2 and b minimize $W(\theta)$, let us put to 0 the derivatives of $W(\theta)$ w.r.t. θ:

$$\begin{aligned} \frac{\partial W(\theta)}{\partial a_1} &= -2(0.5 - a_1) - a_2 = 0 \\ \frac{\partial W(\theta)}{\partial a_2} &= 2a_2 + (0.5 - a_1) = 0 \\ \frac{\partial W(\theta)}{\partial b} &= -(1-b) = 0 \end{aligned}$$

Solving the 3 equations, one gets the values $a_1 = 0.5$, $a_2 = 0$ and $b = 1$, i.e., the values that minimize $\lim_{N\to\infty} V_N(\theta)$ are the exact values for all 3 parameters (in fact the model is unbiased).

(c) The closed loop system is

- for $m = 0$:
$$y(t) = (a_1 - kb)y(t-1) + a_2 y(t-2) + e(t)$$
which is not identifiable (a_1 and b cannot be identified separately)
- for $m = 1$:
$$y(t) = a_1 y(t-1) + (a_2 - kb)y(t-2) + e(t)$$
which is not identifiable (a_2 and b cannot be identified separately)

- for $m > 1$:

$$y(t) = a_1 y(t-1) + a_2 y(t-2) - kby(t-m) + e(t)$$

which is identifiable.

3. Exercise 3: System identification.

   (a) The frequency function (Fig. 1) shows a resonance peak at around 12 rad/sec.

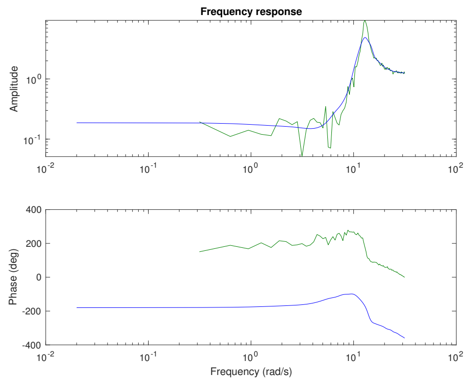

Figure 1: Frequency function.

   (b) The ARX order selection tool of the System Id toolbox suggests ARX(10,10,2) (fit of 99% to estimation data). The number of parameters (20) is however very high (leading for example to high overlap in zeros-poles confidence intervals). Lower order models exist of similar performance. For instance a ARMAX(4,3,2,1) gives a fit of 99.14% (99.1% to estimation data) with much less parameters.

```
amx4321 =
Discrete-time ARMAX model:  A(z)y(t) = B(z)u(t) + C(z)e(t)

  A(z) = 1 - 0.4176 (+/- 0.003234) z^-1 + 0.8691 (+/- 0.0008924) z^-2
  + 0.09399 (+/- 0.002148) z^-3 + 0.02667 (+/- 0.001632) z^-4


  B(z) = -0.9849 (+/- 0.0009865) z^-1 + 1.272 (+/- 0.002975) z^-2
              - 0.5589 (+/- 0.003486) z^-3

  C(z) = 1 - 0.4694 (+/- 0.02159) z^-1 + 0.8877 (+/- 0.02168) z^-2

Name: amx4321
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Sample time: 1 seconds

Parameterization:
   Polynomial orders:   na=4   nb=3   nc=2   nk=1
   Number of free coefficients: 9

Estimated using PEM on time domain data "mydatade".
Fit to estimation data: 99.12\% (prediction focus)
FPE: 0.0004813, MSE: 0.0004642
```

Parameter uncertainty is small. The model fit is shown in violet in Fig. 2. Residuals are in Fig. 3, in violet. Notice that if we reduce the order of the A polynomial from 4 to 3 we have the residuals in blue which show a peak at positive τ, meaning that the order is too low (hence $A(q)$ of order 4 is the least acceptable order). Zero-poles are in Fig. 4 The poles are all stable. Confidence intervals are very

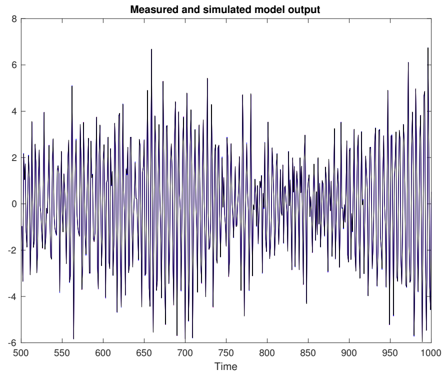

Figure 2: Model fit for BJ(3,2,3,4,2).

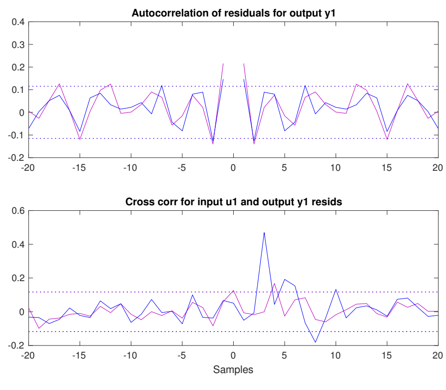

Figure 3: Residuals

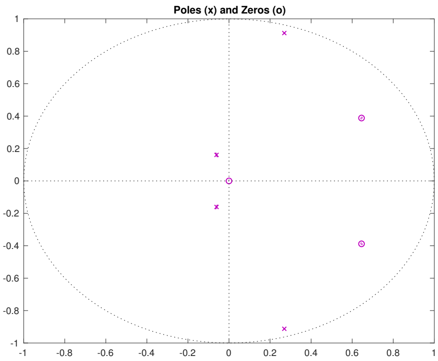

Figure 4: Zero-poles

small. The Bode plot is in Fig. 5. It overlaps well with the frequency functions computed via ETFE and SPA.

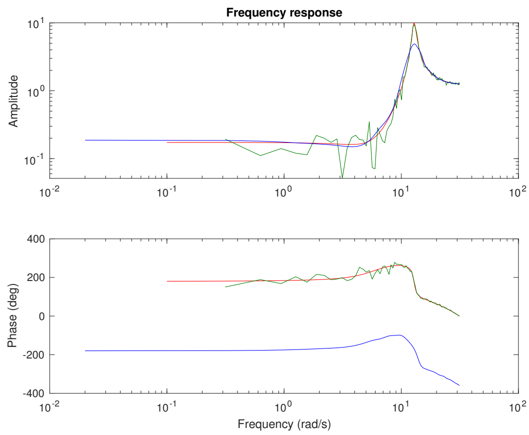

Figure 5: Bode plot and frequency function.

4. Exercise 4: bond graph

   (a) The bond graph is given in Fig. 6 and has no causality conflict. Given the sign of the causality markings, there is no need to invert the nonlinearity $\Delta p = rQ_2^2$.

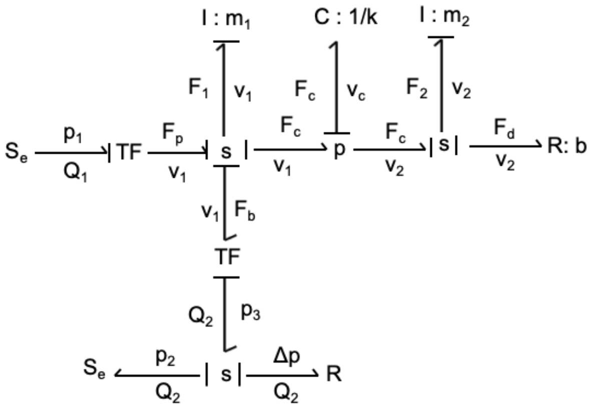

Figure 6: Exercise 4, bond graph

(b) Looking at Fig. 6, we have:

$$\begin{aligned}
\text{TF:} \quad & F_p && = A_1 p_1, \;\; v_1 = Q_1/A_1 \\
\text{s:} \quad & F_p && = F_1 + F_b + F_c \\
\text{TF:} \quad & F_b && = A_2 p_3, \;\; v_1 = Q_2/A_2 \\
\text{s:} \quad & p_3 && = p_2 + \Delta p \\
\text{R:} \quad & \Delta p && = r Q_2^2 \\
\text{I:} \quad & m_1 \dot{v}_1 && = F_1 \\
\text{p:} \quad & v_1 && = v_2 + v_c \\
\text{C:} \quad & \tfrac{1}{k} \dot{F}_c && = v_c \\
\text{s:} \quad & F_c && = F_2 + F_d \\
\text{R:} \quad & F_d && = b v_2 \\
\text{I:} \quad & m_2 \dot{v}_2 && = F_2
\end{aligned}$$

Choosing as state variables v_1, v_2, and F_c and as input p_1 and p_2, one gets the following (nonlinear) state space system

$$\begin{aligned}
\dot{v}_1 &= \frac{1}{m_1} \left(-A_2^3 r v_1^2 - F_c + A_1 p_1 - A_2 p_2 \right) \\
\dot{v}_2 &= \frac{1}{m_2} \left(-b v_2 + F_c \right) \\
\dot{F}_c &= k \left(v_1 - v_2 \right)
\end{aligned}$$

5. Exercise 5

(a) Differentiating the second equation

$$\begin{aligned} 0 &= 2\dot{x}_1 + \dot{x}_1 x_2 + x_1 \dot{x}_2 + \dot{u} \\ &= -2x_1 + 4x_2 - x_1 x_2 + 2x_2^2 + x_1 \dot{x}_2 + \dot{u} \end{aligned}$$

If $x_1 \neq 0$ then one can write

$$\dot{x}_2 = \frac{2x_1 - 4x_2 + x_1 x_2 - 2x_2^2 - \dot{u}}{x_1}$$

and the index is 1. If instead $x_1 = 0$ then one must differentiate again the AE

$$\begin{aligned} 0 &= (-2\dot{x}_1 + 4\dot{x}_2 - \dot{x}_1 x_2 - x_1 \dot{x}_2 + 4x_2 \dot{x}_2 + \dot{x}_1 \dot{x}_2 + x_1 \ddot{x}_2 + \ddot{u})\,|_{x_1=0} \\ &= -4x_2 - 2x_2^2 + (4 + 6x_2)\dot{x}_2 + \ddot{u} \end{aligned}$$

If $x_1 = 0$ and $4 + 6x_2 \neq 0$ (i.e., $x_2 \neq -2/3$) then the differentiability index is 2. The only point in which the differentiability index can be greater than 2 is $(x_1, x_2) = (0, -2/3)$ but we can stop here with the calculations.

(b) Using $u = x_2 - x_1 x_2$, the DAE becomes

$$\begin{aligned} \dot{x}_1 &= -x_1 + 2x_2 \\ 0 &= 2x_1 + x_2 \end{aligned}$$

Differentiating the AE

$$0 = 2\dot{x}_1 + \dot{x}_2 = -2x_1 + 4x_2 + \dot{x}_2$$

or

$$\dot{x}_2 = 2x_1 - 4x_2$$

meaning that the differentiability index is 1 everywhere (a linear system has no singularities when computing the differentiability index).

(c) If one chooses

$$u = \dot{x}_2$$

then the DAE becomes a system of ODEs, hence for it the differentiability index is 0.