Solution for the "Modeling and Simulation" exam (TSRT62) 2019-10-28

1. (a) Choosing $x_1 = y$ and $x_2 = \dot{y}$, one gets

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_2^2 x_1 - x_1^3 + (1 + x_1^2)u$$

(b) Choosing

$$H(q) = \frac{1}{1 + a_1 q^{-1} + a_2 q^{-2}}$$

one gets an ARX model.

(c) Since $R_e(\tau) = \lambda_e \delta(\tau)$ and

$$\begin{aligned} R_y(\tau) &= E\left[y(t)y(t-\tau)\right] \\ &= E\left[(b_1 e(t-1) + \ldots + b_N e(t-N))(b_1 e(t-\tau-1) + \ldots + b_N e(t-\tau-N))\right] \end{aligned}$$

it is $R_y(\tau) = 0$ if $|\tau| > N-1$, i.e., $M = N-1$.

(d) Considering the test function $f(x_n) = \lambda x_n$, the Adams algorithm becomes

$$x_n = x_{n-1} + h\lambda \frac{(x_n + x_{n-1})}{2}$$

or

$$x_n = \frac{1 + \frac{h\lambda}{2}}{1 - \frac{h\lambda}{2}} x_{n-1}$$

The stability condition is

$$\left| \frac{1 + \frac{h\lambda}{2}}{1 - \frac{h\lambda}{2}} \right| < 1 \quad \Longleftrightarrow \quad \left|1 + \frac{h\lambda}{2}\right| < \left|1 - \frac{h\lambda}{2}\right|$$

which is always true for $h\lambda < 0$. Hence for stable systems (i.e., $\lambda < 0$) all sampling intervals h ($h > 0$, obviously) will give a stable Adams method.

2. Denote $b = [b_1\ b_2]^T$.

(a) The prediction error is given by

$$\varepsilon(t, b) = (0.2 - b_1)u(t-1) + (0.4 - b_2)u(t-2) + 0.5u(t-3) + w(t),$$

so that when $N \to \infty$

$$\begin{aligned} V(b) \triangleq \mathrm{E}\,\varepsilon(t,b)^2 &= \Big((0.2 - b_1)^2 + (0.4 - b_2)^2 + 0.25\Big) R_u(0) + \lambda_w \\ &+ 2\left((0.2 - b_1)(0.4 - b_2) + 0.5(0.4 - b_2)\right) R_u(1) + (0.2 - b_1)R_u(2) \\ &+ 2(0.2 - b_1)R_{wu}(1) + 2(0.4 - b_2)R_{wu}(2) + R_{wu}(3). \end{aligned}$$

Since $R_{wu}(k) = 0$ for all k and $R_u(\tau) = \delta(\tau)\lambda_u$, it is

$$V(b) = \left((0.2 - b_1)^2 + (0.4 - b_2)^2 + 0.25\right) \lambda_u + \lambda_w$$

The optimization condition is

$$\frac{\partial V(b)}{\partial b} = 2\lambda_u \begin{bmatrix} 0.2 - b_1 \\ 0.4 - b_2 \end{bmatrix} = 0$$

which is still corresponding to the "true values" $\hat{b}_1 = b_1^o = 0.2$ and $\hat{b}_2 = b_2^o = 0.4$ even though the model is not unbiased (it does not contain the true system for any value of b_i).

(b) In this case the model is unbiased, and a calculation identical to the previous one yields

$$V(b) = \left((0.2 - b_1)^2 + (0.4 - b_2)^2\right) R_u(0) + \lambda_w + 2(0.2 - b_1)(0.4 - b_2)R_u(1) \\ + 2(0.2 - b_1)R_{wu}(1) + 2(0.4 - b_2)R_{wu}(2)$$

which leads again to $\hat{b}_1 = b_1^o = 0.2$ and $\hat{b}_2 = b_2^o = 0.4$. For finite N, the variance of the estimates can be computed using the following formula

$$P_N = \mathrm{Cov}[b - b^o] \simeq \frac{\lambda_w}{N}\bar{R}^{-1}$$

where $\bar{R} = \mathrm{E}\,\phi(t)\phi(t)^T$, $\phi(t) = \begin{bmatrix} u(t-1) \\ u(t-2) \end{bmatrix}$. In particular:

$$\bar{R} = \begin{bmatrix} R_u(0) & R_u(1) \\ R_u(1) & R_u(0) \end{bmatrix} \quad \Longrightarrow \quad \bar{R}^{-1} = \frac{1}{R_u(0)^2 - R_u(1)^2} \begin{bmatrix} R_u(0) & -R_u(1) \\ -R_u(1) & R_u(0) \end{bmatrix}$$

i.e.,

$$P_N \simeq \frac{1}{(1 - 0.25)N} \begin{bmatrix} 1 & -0.5 \\ -0.5 & 1 \end{bmatrix}$$

meaning that $\mathrm{Var}[b_i - b_i^o] = \frac{4}{3N}$.

(c) One possible input signal is a white noise with variance $\lambda_u > 0.75$. In fact, for it $R_u(0) = \lambda_u$, while $R_u(1) = 0$, so

$$P_N \simeq \frac{1}{\lambda_u^2 N} \begin{bmatrix} \lambda_u & 0 \\ 0 & \lambda_u \end{bmatrix} \quad \Longrightarrow \quad \mathrm{Var}[b_i - b_i^o] = \frac{1}{\lambda_u N}$$

Another possibility is for instance to take $u(t)$ such that $R_u(0) > 1$, and $R_u(\tau)$ is the same as in (b) for $\tau \neq 0$.

3. Exercise 3: System identification.

   (a) The frequency functions estimated with both ETFE and SPA are basically identical and are reported in Fig. 1. They show two resonant peaks, at 1 rad/s and at 2 rad/s. Hence model (a) is a suitable model. However, since $8.28 - 2\pi = 2$ and also $14.56 - 4\pi = 2$, also (b) and (c) could have generated the data because of aliasing.

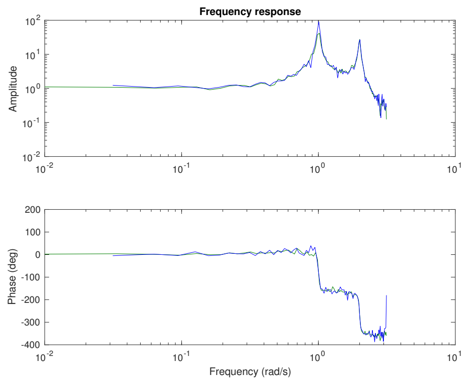

Figure 1: ETFE and SPA for the data.

   (b) The ARX order selection tool of the System Id toolbox suggests ARX(8,10,1) (fit of 98.9% to estimation data). The number of parameters (18) is however very high (leading for example to high overlap in zeros-poles confidence intervals). Lower order models exist of similar performance. For instance a BJ(4,1,1,4,1) gives a fit of 98.99% (98.68% to estimation data) with much less parameters.

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bj41141 =
Discrete-time BJ model:  y(t) = [B(z)/F(z)]u(t) + [C(z)/D(z)]e(t)

  B(z) = 0.6482 (+/- 0.002381) z^-1 + 2.17 (+/- 0.002377) z^-2
           + 0.07146 (+/  - 0.002565) z^-3 - 0.3572 (+/- 0.002553) z^-4

  C(z) = 1 + 0.1366 (+/- 0.5996) z^-1

  D(z) = 1 + 0.06244 (+/- 0.604) z^-1


  F(z) = 1 - 0.2543 (+/- 8.859e-05) z^-1 + 1.068 (+/- 9.538e-05) z^-2
            - 0.2285 (  +/- 9.074e-05) z^-3 + 0.9417 (+/- 8.302e-05) z^-4

Name: bj41141
Sample time: 1 seconds

Parameterization:
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```
   Polynomial orders:   nb=4   nc=1   nd=1   nf=4   nk=1
   Number of free coefficients: 10

Estimated using PEM on time domain data "mydatade".
Fit to estimation data: 98.68\% (prediction focus)
FPE: 0.0104, MSE: 0.009994
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Parameter uncertainty is small for the input-output transfer function, but it is quite big for the noise-output transfer function (nobody is perfect... but it is a sign that the noise needs no model, hence a simpler model like OE(4,4,1) would have done a better job in this case). The model fit is shown in Fig. 2. Residuals

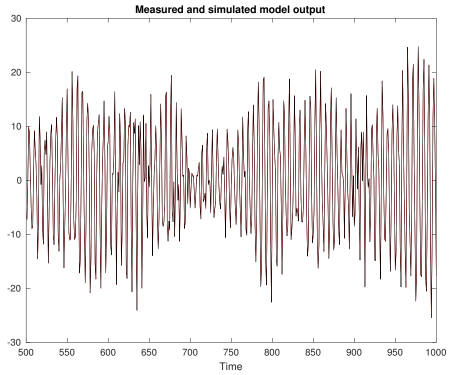


Figure 2: Model fit for BJ(4,1,1,4,1).

and zeros/poles are in Fig. 3. The poles are all stable. Confidence intervals are

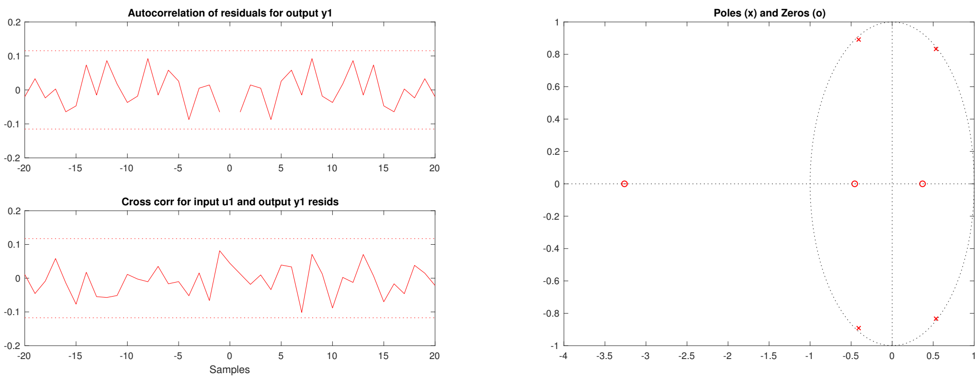


Figure 3: Residuals and zero-poles

very very small. The Bode plot is in Fig. 4 (red). It overlaps very well with the frequency functions computed via ETFE and SPA.

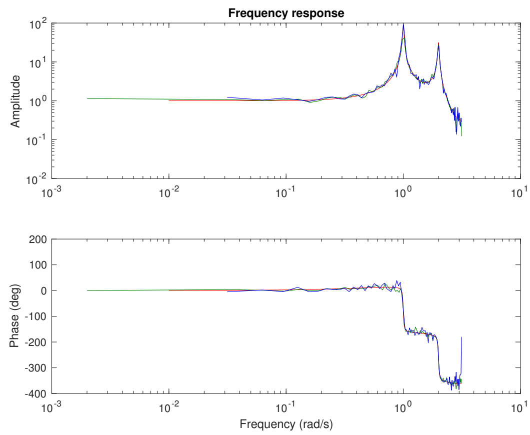


Figure 4: Bode plot.

4. Exercise 4: bond graph

   (a) The bond graph is given in Fig. 5 and has no causality conflict. Given the sign of the causality markings, there is no need to invert $\phi(\cdot)$.

   (b) When the masses are fixed in pairs, then their velocity differences Δv_i become 0, hence the p-junctions disappear (as well as the two R-type elements) and we obtain the graph of Fig. 6, which shows two causality conflicts. These conflicts can be eliminated by "merging" the masses, as in the conflict-free graph of Fig. 7.

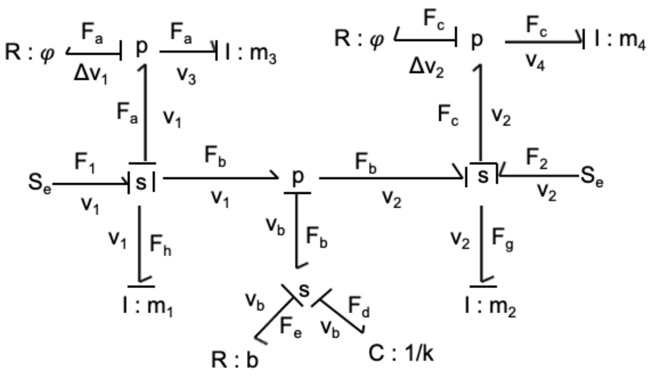


Figure 5: Exercise 4, original bond graph

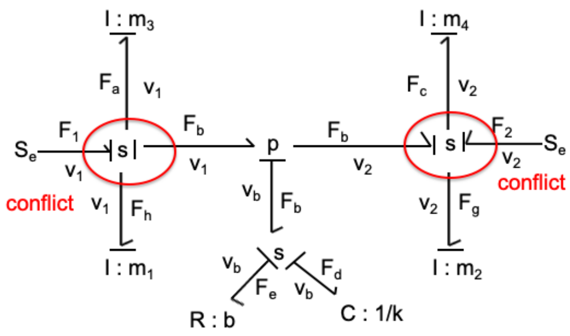


Figure 6: Exercise 4, bond graph with connected masses

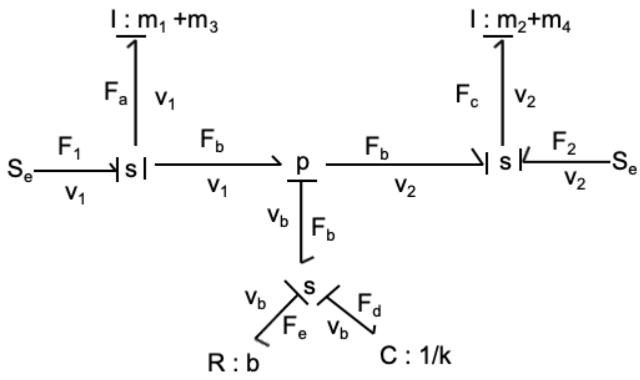


Figure 7: Exercise 4, bond graph simplified

(c) Looking at Fig. 7, we have:

$$
\begin{aligned}
\text{s:} &\quad F_1 &&= F_a + F_b \\
\text{I:} &\quad (m_1 + m_3)\dot{v}_1 &&= F_a \\
\text{p:} &\quad v_1 &&= v_2 + v_b \\
\text{s:} &\quad F_b &&= F_d + F_e \\
\text{C:} &\quad \frac{1}{k}\dot{F}_d &&= v_b \\
\text{R:} &\quad F_e &&= b v_b \\
\text{s:} &\quad F_b + F_2 &&= F_c \\
\text{I:} &\quad (m_2 + m_4)\dot{v}_2 &&= F_c
\end{aligned}
$$

Choosing as state vector $x = [v_1\ v_2\ F_d]^T$ and as input $u = [F_1\ F_2]^T$, one gets the following state space system

$$\dot{x} = \begin{bmatrix} -\frac{b}{m_1+m_3} & \frac{b}{m_1+m_3} & -\frac{1}{m_1+m_3} \\ \frac{b}{m_2+m_4} & -\frac{b}{m_2+m_4} & \frac{1}{m_2+m_4} \\ k & -k & 0 \end{bmatrix} x + \begin{bmatrix} \frac{1}{m_1+m_3} & 0 \\ 0 & \frac{1}{m_2+m_4} \\ 0 & 0 \end{bmatrix} u$$

5. Exercise 5: DAE

(a) The DAE for the circuit are

$$\begin{aligned} L_1\frac{di_1}{dt} - v &= 0 \\ L_2\frac{di_2}{dt} - v + Ri_2 &= 0 \\ i_1 + i_2 &= i \end{aligned}$$

or, calling $z = [i_1\ i_2\ v]^T$ the state vector,

$$E\dot{z} + Fz = \begin{bmatrix} L_1 & 0 & 0 \\ 0 & L_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{z} + \begin{bmatrix} 0 & 0 & -1 \\ 0 & R & -1 \\ 1 & 1 & 0 \end{bmatrix} z = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} i$$

(b) The matrix E is not full rank. Differentiating the last row:

$$\begin{bmatrix} L_1 & 0 & 0 \\ 0 & L_2 & 0 \\ 1 & 1 & 0 \end{bmatrix} \dot{z} + \begin{bmatrix} 0 & 0 & -1 \\ 0 & R & -1 \\ 0 & 0 & 0 \end{bmatrix} z = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \frac{di}{dt}$$

which is still not full rank. Rearranging and differentiating again the last row

$$\begin{bmatrix} L_1 & 0 & 0 \\ 0 & L_2 & 0 \\ 0 & -\frac{R}{L_2} & \frac{L_1+L_2}{L_1L_2} \end{bmatrix} \dot{z} + \begin{bmatrix} 0 & 0 & -1 \\ 0 & R & -1 \\ 0 & 0 & 0 \end{bmatrix} z = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \frac{d^2i}{dt^2}$$

Now the matrix in front of $\dot{z}$ is full rank, hence the differentiability index is 2.

(c) From $w = L_1 i_1 - L_2 i_2$, we get

$$\dot{w} = L_1\frac{di_1}{dt} - L_2\frac{di_2}{dt} = Ri_2 \tag{S1}$$

Combining the following two equations

$$w = L_1 i_1 - L_2 i_2 \qquad i = i_1 + i_2$$

we get

$$i_1 = \frac{w + L_2 i}{L_1 + L_2}, \qquad i_2 = \frac{L_1 i - w}{L_1 + L_2}$$

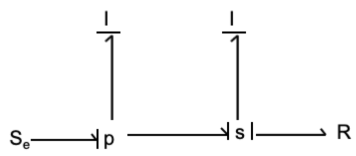
which inserted in (S1) gives

$$\dot{w} = -\frac{R}{L_1 + L_2} w + \frac{R L_1}{L_1 + L_2} i$$

with output

$$y = v = L_1 \frac{di_1}{dt} = -\frac{R L_1}{(L_1 + L_2)^2} w + \frac{R L_1^2}{(L_1 + L_2)^2} i + \frac{L_1 L_2}{L_1 + L_2} \frac{di}{dt}$$

(d) Replacing the current source with voltage source, the bond graph is



which has conflict-free causality, hence the new system can be transformed into state space form.