Solution for the “Modeling and Simulation" exam (TSRT62) 2019-08-26

1. (a) Equilibrium points are computed as solutions of

$$0 = 2x_1 - x_1x_2 = x_1(2 - x_2),$$
$$0 = 2x_1^2 - x_2.$$

From the first equation $x_1 = 0$ or $x_2 = 2$. With $x_1 = 0$ the second equation gives $x_2 = 0$. With $x_2 = 2$ the second equation gives $x_1 = \pm 1$. Therefore the three equilibrium points are $(x_1, x_2) = (0, 0), (-1, 2), (1, 2)$.

(b) The model is an OE model hence this class gives a nonbiased predictor. Of course one can also use "degenerate" ARMAX (with full zero/pole cancelation: $A(q) = D(q)$ and also $C(q) = D(q)$ (so that $\frac{C(q)}{A(q)} = 1$ in the noise to output channel) and, in the same spirit, a "degenerate" BJ model.

(c) It is possible to rescale the system

$$\ddot{y} + 2\dot{y} + 5y = -\dot{u} + u \tag{S1}$$

to make it be exactly equal to

$$4\ddot{y} + 4\dot{y} + 5y = -6\dot{u} + 3u \tag{S2}$$

To do so, define

$$\tau = \beta t, \qquad z = \alpha y \quad \text{and} \quad v = u$$

Then

$$y(t) = \frac{1}{\alpha} z(\tau) = \frac{1}{\alpha} z(\beta t)$$

Differentiating

$$z' = \frac{dz}{d\tau} = \alpha \frac{dy(t)}{d\tau} = \alpha \frac{dy}{dt}\frac{dt}{d\tau} = \frac{\alpha}{\beta}\dot{y}$$

$$z'' = \frac{d^2 z}{d\tau^2} = \frac{d}{d\tau}\left(\frac{dz}{d\tau}\right) = \frac{\alpha}{\beta}\frac{d}{d\tau}\dot{y} = \frac{\alpha}{\beta}\ddot{y}\frac{dt}{d\tau} = \frac{\alpha}{\beta^2}\ddot{y}$$

and

$$v' = \frac{dv}{d\tau} = \frac{du}{dt}\frac{dt}{d\tau} = \frac{\dot{u}}{\beta}$$

In the new variables, the system (S1) becomes

$$\frac{\beta^2}{\alpha} z'' + \frac{2\beta}{\alpha} z' + \frac{5}{\alpha} z = -\beta v' + v$$

Multiply both sides by 3 and equate each term with the second system (S2), which in the new variables and with the new notation for differentiation is

$$4z'' + 4z' + 5z = -6v' + 3v$$

This corresponds to choosing

$$\beta = 2 \quad \text{and} \quad \alpha = 3$$

The step response of (S2) therefore corresponds to rescaling the time axis by 2 and the vertical axis by 3, see Fig. 1.

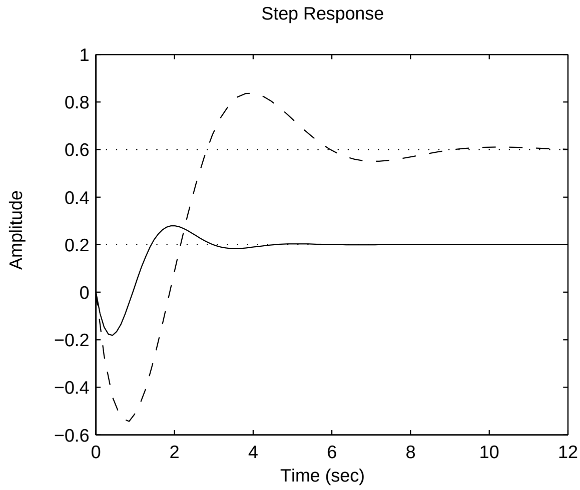
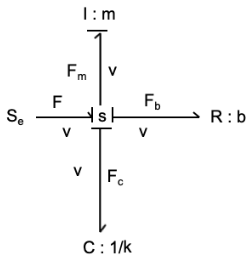
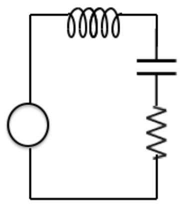


Figure 1:

(d) The bond graph and equivalent electrical circuits are





The correspondence is obvious.

2. The prediction error is given by

$$\varepsilon(t,b) = (1-b_1)u(t-1) + (0.7-b_2)u(t-2) + e(t),$$

so that

$$\begin{aligned} V(b) \triangleq \mathrm{E}\,\varepsilon(t,b)^2 &= (1-b_1)^2 R_u(0) + (0.7-b_2)^2 R_u(0) + 2(1-b_1)(0.7-b_2)R_u(1) \\ &\quad + R_e(0) + 2(1-b_1)R_{ue}(1) + 2(0.7-b_2)R_{ue}(2). \end{aligned}$$

(a) Since $R_{ue}(k) = 0$ for $k \geq 0$ plus $R_u(0) = 1$ and $R_u(1) = 1/2$, it is

$$V(b) = (1-b_1)^2 + (0.7-b_2)^2 + (1-b_1)(0.7-b_2) + 1.$$

The optimization conditions are then

$$0 = \nabla V(b) = \begin{pmatrix} 2b_1 + b_2 - 2.7 \\ 2b_2 + b_1 - 2.4 \end{pmatrix},$$

which gives $\widehat{b} = (1, 0.7)$. Since $\nabla^2 V(\widehat{b}) \geq 0$, $\widehat{b}$ is indeed a minimum. This shows that the calculations are consistent even when the noise $e(t)$ is not white (provided it is uncorrelated with the input u).

(b) In this case it is $R_{ue}(1) = 1/5$ and $R_{ue}(2) = 1/25$, so that

$$V(b) = (1 - b_1)^2 + (0.7 - b_2)^2 + (1 - b_1)(0.7 - b_2) + 1 \\ + 2(1 - b_1)/5 + 2(0.7 - b_2)/25.$$

This leads to the optimality conditions

$$0 = \nabla V(b) = \begin{pmatrix} 2b_1 + b_2 - 3.1 \\ b_1 + 2b_2 - 2.48 \end{pmatrix},$$

which gives $\widehat{b} = (1.24, 0.62)$, for which $\nabla^2 V(\widehat{b}) \geq 0$, i.e., $\widehat{b}$ is again a minimum. This shows that the calculations can become biased when the noise is correlated with the input u.

3. Exercise 3: System identification.

(a) The frequency functions estimated with both ETFE and SPA (with $M = 1000$) are very similar and are reported in Fig. 2. They show a resonant peak at 1 rad/s.

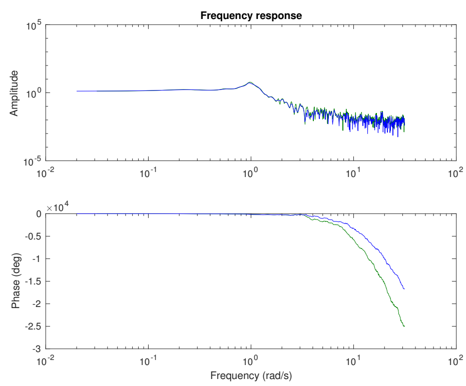


Figure 2: ETFE and SPA for the data.

(b) The ARX order selection tool of the System Id toolbox suggests ARX(9,9,2) (fit of 73.31% to estimation data). The number of parameters (18) is however very high (leading for example to high overlap in zeros-poles confidence intervals). Lower order models exist of even better performance. For instance a BJ(3,2,3,4,2) gives a fit of 90.23% (99.5% to estimation data!) with much less parameters.

```
bj32342 =
Discrete-time BJ model:  y(t) = [B(z)/F(z)]u(t) + [C(z)/D(z)]e(t)

  B(z) = 0.001279 (+/- 8.591e-05) z^-2 + 0.001175 (+/- 7.316e-05) z^-3
             - 0.0006878 (+/- 7.246e-05) z^-4

  C(z) = 1 - 0.2508 (+/- 0.0326) z^-1 - 0.7306 (+/- 0.03128) z^-2


  D(z) = 1 - 2.815 (+/- 0.03547) z^-1 + 2.652 (+/- 0.06985) z^
              -2 - 0.8361 (+/- 0.03505) z^-3



  F(z) = 1 - 2.394 (+/- 0.03124) z^-1 + 1.355 (+/- 0.09037) z^-2
             + 0.4898 (+ /- 0.08838) z^-3 - 0.4501 (+/- 0.02922) z^-4

Name: bj32342
Sample time: 0.1 seconds

Parameterization:
   Polynomial orders:   nb=3   nc=2   nd=3   nf=4   nk=2
   Number of free coefficients: 12
   Use "polydata", "getpvec", "getcov" for parameters and their uncertainties.

Status:
Termination condition: Maximum number of iterations reached.
Number of iterations: 20, Number of function evaluations: 545

Estimated using PEM on time domain data "mydatade".
Fit to estimation data: 99.56% (prediction focus)
FPE: 6.477e-06, MSE: 6.003e-06
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Parameter uncertainty is small. The model fit is shown in violet in Fig. 3. Residuals and zeros/poles are in Fig. 4. The poles are all stable. Confidence intervals are very small. The Bode plot is in Fig. 5. It overlaps well with the frequency functions computed via ETFE and SPA.

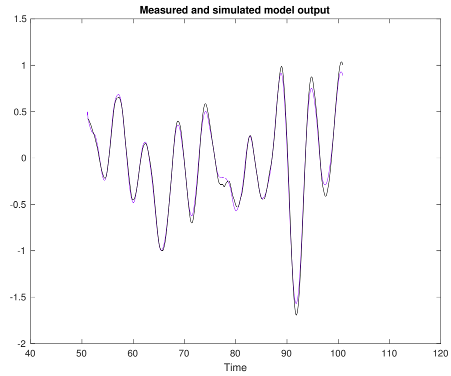

Figure 3: Model fit for BJ(3,2,3,4,2).

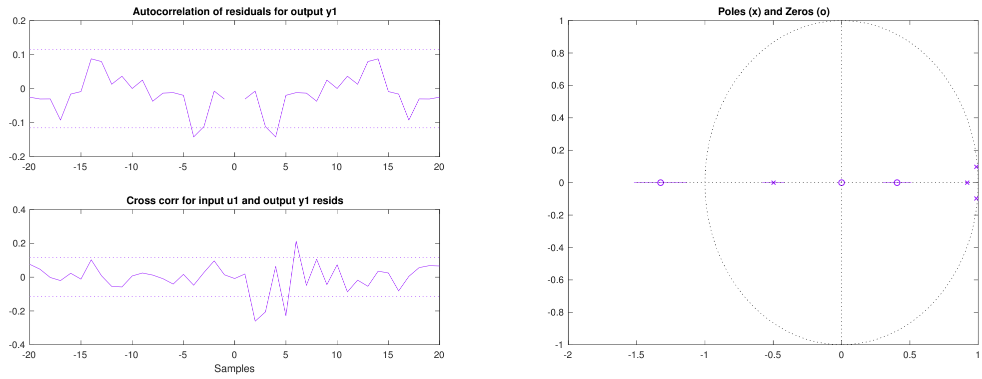

Figure 4: Residuals and zero-poles

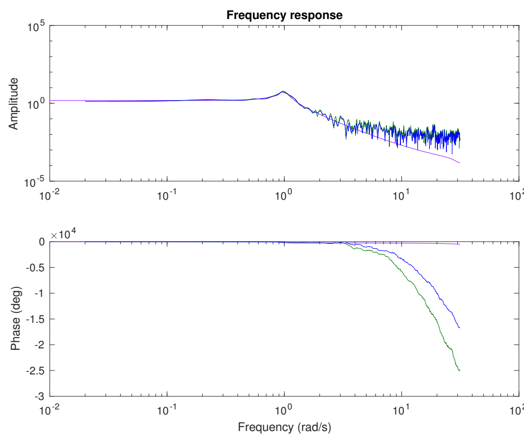

Figure 5: Bode plot.

4. Exercise 4: bond graph

   (a) The bond graph is given in Fig. 6. As shown, there is indeed a causality conflict. However, by exchanging TF and the s-junction to its right we obtain the graph of Fig. 7, which can be simplified to the conflict-free graph of Fig. 8, where the two I-type bonds have been merged. The elimination of the TF implies that we re-

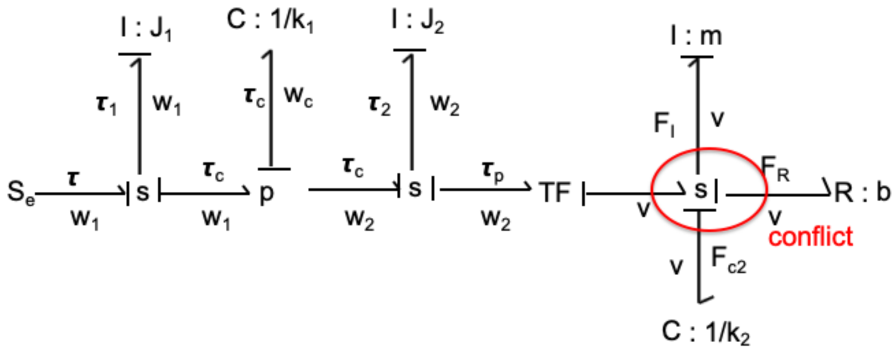

Figure 6: Exercise 4, original bond graph

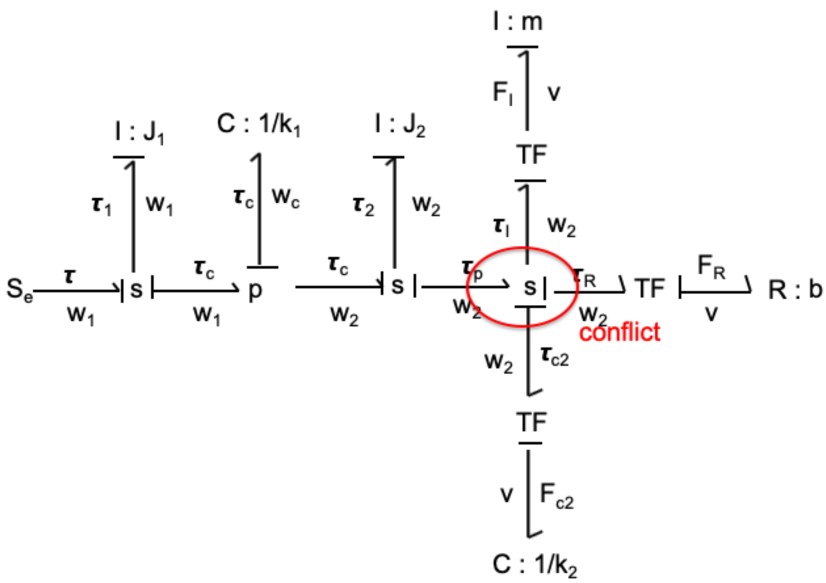

Figure 7: Exercise 4, bond graph rearranged

place the linear mechanical variables with the corresponding rotational equivalent,

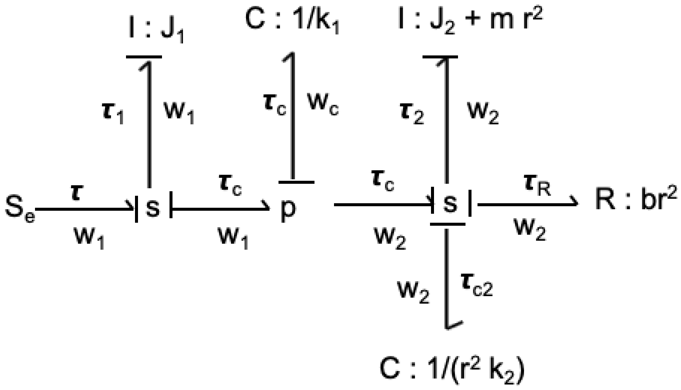

Figure 8: Exercise 4, bond graph simplified

according to the following formulas:

$$v = rw_2, \qquad F_I = \frac{\tau_I}{r}, \qquad F_{C2} = \frac{\tau_{C2}}{r}, \qquad F_R = \frac{\tau_R}{r}$$

(b) Looking at Fig. 8, we have:

$$\begin{array}{rcl}
\text{s:} & \tau & = \tau_1 + \tau_C \\
\text{I:} & J_1 \frac{dw_1}{dt} & = \tau_1 \\
\text{p:} & w_1 & = w_2 + w_C \\
\text{C:} & \frac{1}{k_1} \frac{d\tau_C}{dt} & = w_C \\
\text{s:} & \tau_C & = \tau_2 + \tau_{C2} + \tau_R \\
\text{R:} & \tau_R & = br^2 w_2 \\
\text{I:} & (J_2 + mr^2) \frac{dw_2}{dt} & = \tau_2 \\
\text{C:} & \frac{1}{r^2 k_2} \frac{d\tau_{C2}}{dt} & = w_2
\end{array}$$

Choosing as state vector $x = [w_1 \; w_2 \; \tau_C \; \tau_{C2}]^T$, one gets the following system of ODEs

$$\frac{dx}{dt} = \begin{bmatrix} 0 & & -\frac{1}{J_1} & 0 & \\ k_1 & -k_1 & 0 & 0 & \\ 0 & -\frac{br^2}{J_2+mr^2} & 0 & \frac{1}{J_2+mr^2} & -\frac{1}{J_2+mr^2} \\ 0 & k_2 r^2 & 0 & 0 & \end{bmatrix} x + \begin{bmatrix} \frac{1}{J_1} \\ 0 \\ 0 \\ 0 \end{bmatrix} \tau$$

5. Exercise 5

(a) Differentiating equation (3) yields:

$$\dot{x}_1 x_3 + x_1 \dot{x}_3 - \dot{x}_2(2 + x_3) - x_2 \dot{x}_3 + 1 = 0$$

or

$$(1 + x_2)x_3 - \dot{x}_2(2 + x_3) + 1 = 0 \tag{S3}$$

Since $\dot{x}_3 = 0$, it is $x_3(t) = x_3(0) = \text{const}$ for all t.

i. If $x_3 \neq -2$, then one can write

$$\dot{x}_2 = \frac{1 + x_3 + x_2 x_3}{2 + x_3} \tag{S4}$$

and hence the index is 1

ii. If instead $x_3 = -2$ then (S3) becomes

$$x_3 + x_2 x_3 + 1 = 0 \tag{S5}$$

which can be differentiated again, leading to

$$-2\dot{x}_2 = 0$$

and therefore the index is 2 in this case.

(b) From the algebraic constraint (4) we have, differentiating,

$$\dot{x}_1 x_3 + x_1 \dot{x}_3 - 2\dot{x}_2 + 1 = 0$$

or

$$\dot{x}_2 = \frac{1}{2} + \frac{(1 + x_2)x_3}{2}$$

In this case the index is =1 always, without singular points.

(c) Split into two cases:

i. When $x_2 \neq -2$ we have an explicit expression for x_2 from (3)

$$x_2 = \frac{x_1 x_3 + t}{2 + x_3} \tag{S6}$$

This can be used to eliminate the dependence from x_2 in eq. (1), hence obtaining

$$\begin{aligned} \dot{x}_1 &= 1 + \frac{x_1 x_3 + t}{2 + x_3} && \text{(S7)} \\ \dot{x}_3 &= 0 && \text{(S8)} \end{aligned}$$

The system (S7)-(S8) can be solved independently of x_2. Since (S8) corresponds to $x_3(t) = x_3(0) = \text{const}$, only (S7) needs to be actually solved. Then (S6) can be used to obtain x_2 from the solution of this ODE.

ii. When $x_3 = -2$ the previous method does not work. However, $x_3 = -2$, and from (S5) and $\dot{x}_2 = 0$,

$$x_2(t) = x_2(0) = -\frac{1}{2}$$

which implies

$$\begin{aligned} \dot{x}_1 &= \frac{1}{2} \\ x_2(t) &= -\frac{1}{2} \qquad \forall\, t \\ x_3(t) &= -2 \qquad \forall\, t \end{aligned}$$