Solution for the "Modeling and Simulation" exam (TSRT62) 2019-01-10

1. (a) Three possible reasons are
   i. Bad parametrization (ex. in $y(t) = (a+b)u(t-1) + e(t)$ cannot distinguish between a and b);
   ii. Bad choice of input (ex. concentrated in too few frequencies);
   iii. Presence of a feedback loop.

   (b) Alice with a program for solving linear systems of equations can only use ARX, as this class of models is linear in the parameters. Bob can use all classes of models.

   (c) For stationary stochastic processes, since u and e are uncorrelated, the spectrum can be computed as

$$\begin{aligned}\Phi_y(\omega) &= |G(i\omega)|^2\Phi_u(\omega) + \Phi_e(\omega) = G(i\omega)G(-i\omega)\Phi_u(\omega) + \Phi_e(\omega) \\ &= \frac{\omega^2+\alpha^2}{\omega^2+\beta^2} + 2 = \frac{3\omega^2+\alpha^2+2\beta^2}{\omega^2+\beta^2}\end{aligned}$$

   (d) The system is asymptotically stable, hence at equilibrium it is $\dot{x} = 0$, from which in correspondence of the input u_o, we get the stationary value $x_o = -\frac{u_o^2}{3}$. Hence the input-output static relationship is $y_o = x_o^2 = \frac{u_o^4}{9}$.

   (e) For a linear ODE $\dot{x} = \lambda x$, the forward Euler method is stable if $|1 + h\lambda| < 1$. In this case, the most restrictive eigenvalue is $\lambda = -4$, hence

$$|1-4h| < 1 \qquad \Longrightarrow \qquad 0 < h < \frac{1}{2}$$

2. Exercise 2. With the given generating expression for u, when $N \to \infty$, the autocorrelation of u is given by

$$\begin{aligned}R_u(0) &= E[u^2(t)] = E[w(t)+w(t-1)]^2 = R_w(0) + 2R_w(1) + R_w(0) = \frac{1}{2} + 0 + \frac{1}{2} = 1 \\ R_u(1) &= E[u(t)u(t-1)] = E[(w(t)+w(t-1))(w(t-1)+w(t-2)] \\ &= R_w(0) + 2R_w(1) + R_w(2) = \frac{1}{2} \\ R_u(\tau) &= 0 \qquad \tau \geq 2\end{aligned}$$

   For the prediction error we have:

$$\bar{V}(\theta) = \lim_{N\to\infty} V_N(\theta) = E[y(t) - \hat{y}(t|\theta)]^2 = E[0.5u(t-1) + 0.7u(t-2) + v(t) - bu(t-k)]^2$$

- for $k = 1$:

$$\begin{aligned}\bar{V}(\theta) =& E[(0.5-b)u(t-1) + 0.7u(t-2) + v(t)]^2 \\ =& (0.5-b)^2 R_u(0) + 0.7^2 R_u(0) + \lambda_v + 2\cdot 0.7\cdot(0.5-b)R_u(1) + 0 \\ =& (0.5-b)^2 + 0.7^2 + 0.7\cdot(0.5-b) + \lambda_v\end{aligned}$$

  Hence

$$\frac{\partial \bar{V}}{\partial b} = -1 - 0.7 + 2b = 0 \quad \Longrightarrow \quad b = 0.85$$

- for $k = 2$, analogously,

$$\bar{V}(\theta) = 0.25 + (0.7-b)^2 + 0.5(0.7-b) + \lambda_v$$

  hence

$$\frac{\partial \bar{V}}{\partial b} = -1.9 + 2b = 0 \quad \Longrightarrow \quad b = 0.95$$

- for $k = 3$,

$$\bar{V}(\theta) = 0.25 + 0.49 + b^2 + \lambda_v + 0.35 - 0.7b$$

  hence

$$\frac{\partial \bar{V}}{\partial b} = 2b - 0.7 = 0 \quad \Longrightarrow \quad b = 0.35$$

- for $k > 3$,

$$\bar{V}(\theta) = 0.25 + 0.49 + b^2 + \lambda_v + 0.35$$

  hence

$$\frac{\partial \bar{V}}{\partial b} = 2b = 0 \quad \Longrightarrow \quad b = 0$$

3. Exercise 3: System identification.

   (a) All 4 classes of models give good fit ($> 75\%$). The one shown here is BJ(3,1,1,3,1) which has the highest fit (83.9% for validation data), although its residuals plot is not perfect.

```
bj31131 =
Discrete-time BJ model:  y(t) = [B(z)/F(z)]u(t) + [C(z)/D(z)]e(t)
  B(z) = 0.259 (+/- 0.01631) z^-1 + 0.4954 (+/- 0.02223) z^-2
            - 0.03493 (+/- 0.02454) z^-3

  C(z) = 1 + 0.8684 (+/- 0.04612) z^-1

  D(z) = 1 + 0.9532 (+/- 0.02832) z^-1
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```
  F(z) = 1 - 0.6586 (+/- 0.01031) z^-1 + 0.9263 (+/- 0.002202) z^-2
              - 0.6888 (+/- 0.009323) z^-3

Name: bj31131
Sample time: 1 seconds

Parameterization:
   Polynomial orders:   nb=3   nc=1   nd=1   nf=3   nk=1
   Number of free coefficients: 8

Status:
Termination condition: Near (local) minimum, (norm(g) < tol).
Number of iterations: 5, Number of function evaluations: 11

Estimated using PEM on time domain data "mydatade".
Fit to estimation data: 86.82\% (prediction focus)
FPE: 0.02728, MSE: 0.02642
```

Parameter uncertainty is small. The model fit is shown in blue in Fig. 1. Residuals

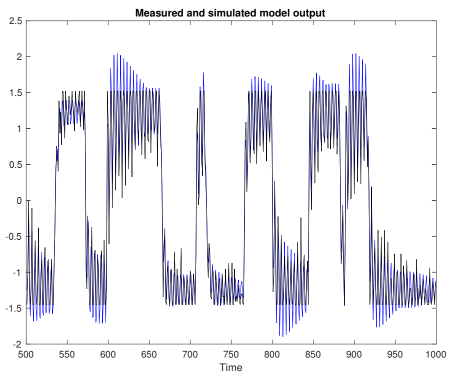

Figure 1: Model fit for BJ(3,1,1,3,1).

and zeros/poles are in Fig. 2. The poles are all stable. Confidence interval are very small. The Bode plot is in Fig. 3.

(b) As can be seen by looking at y (time series in black in Fig. 1), the output of the system shows a saturation. This can be modeled using a Hammerstein-Wiener model (actually a Wiener model would be enough). Selecting a model with linear part having 4 poles and 4 zeros, and with e.g. 3 cut points in both input and output nonlinearity, one gets a fit of 88.22%, see Fig. 4.

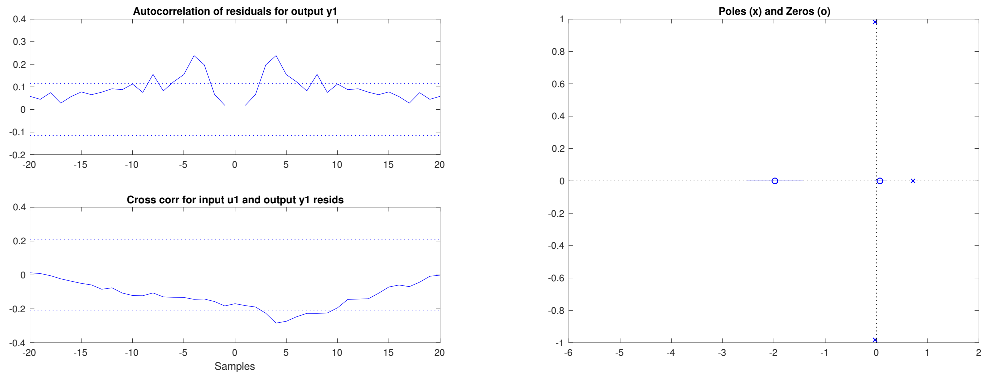

Figure 2: Residuals and zero-poles

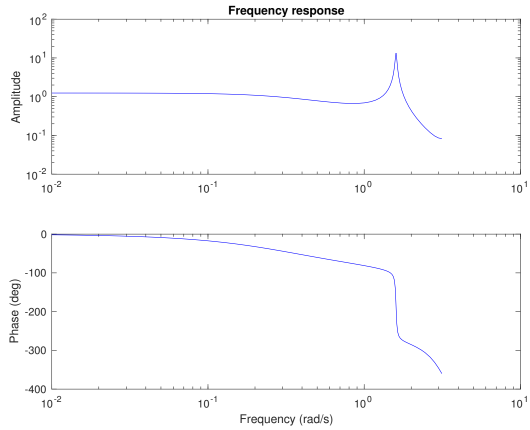

Figure 3: Bode plot.

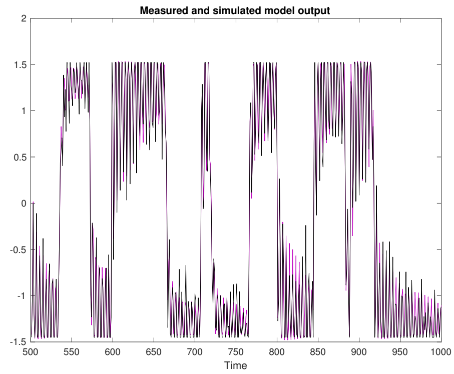

Figure 4: Hammerstein-Wiener model (violet, poorly visible...).

4. Exercise 4: bond graph

(a) The bond graph is given in Fig. 5 and its causality is conflict-free.

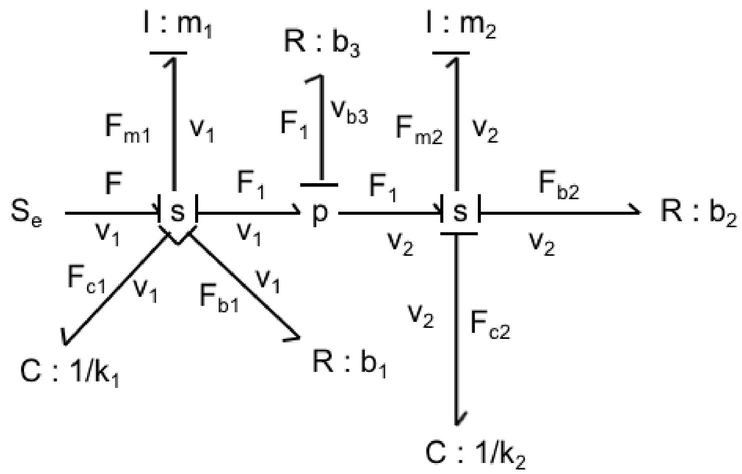

Figure 5: Exercise 4, bond graph

(b) With the choice of variable names shown in Fig. 5 one gets

$$
\begin{array}{rcl}
\text{s:} & F & = F_{m_1} + F_1 + F_{b_1} + F_{c_1} \\
\text{I:} & m_1 \frac{dv_1}{dt} & = F_{m_1} \\
\text{C:} & \frac{1}{k_1}\frac{dF_{c_1}}{dt} & = v_1 \\
\text{R:} & F_{b_1} & = b_1 v_1 \\
\text{p:} & v_1 & = v_2 + v_{b_3} \\
\text{R:} & F_1 & = b_3 v_{b_3} \\
\text{s:} & F_1 & = F_{m_2} + F_{b_2} + F_{c_2} \\
\text{I:} & m_2 \frac{dv_2}{dt} & = F_{m_2} \\
\text{R:} & F_{b_2} & = b_2 v_2 \\
\text{C:} & \frac{1}{k_2}\frac{dF_{c_2}}{dt} & = v_2
\end{array}
$$

Choosing as state vector $x = [v_1 \ F_{c_1} \ v_2 \ F_{c_2}]^T$, one gets the following system of ODEs

$$
\frac{dx}{dt} = \begin{bmatrix} -\frac{b_1+b_3}{m_1} & -\frac{1}{m_1} & \frac{b_3}{m_1} & 0 \\ k_1 & 0 & 0 & 0 \\ \frac{b_3}{m_2} & 0 & -\frac{b_2+b_3}{m_2} & -\frac{1}{m_2} \\ 0 & 0 & k_2 & 0 \end{bmatrix} x + \begin{bmatrix} \frac{1}{m_1} \\ 0 \\ 0 \\ 0 \end{bmatrix} F
$$

(c) Given the direction of the causality markings, any of the 3 dampers can be replaced by nonlinear, noninvertible elements.

5. Exercise 5. Denote

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad F = \begin{bmatrix} 0 & -a \\ 1 & a \end{bmatrix}$$

(a) Solvability corresponds to $\det(sE + F)$ being non-identically zero for s which is not a root (i.e., a pole of the system). Computing

$$\det(sE + F) = \det \begin{bmatrix} s & -a \\ 1 & a \end{bmatrix} = a(s + 1)$$

hence for $a \neq 0$ the system is uniquely solvable, and $s = -1$ is its pole.

(b) With the suggested change of basis we have $w_1 = z_1$, and $az_2 = w_2 - z_1 = w_2 - w_1$, hence

$$\dot{z}_1 + az_2 = 0 \quad \Longrightarrow \quad \dot{w}_1 - (w_2 - w_1) = 0$$

or, in vector form,

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \dot{w} + \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} w = \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

Adding the second row to the first one, we get the “standard form I”

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \dot{w} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} w = \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

(c) Since in the “standard form I” it is $N = 0$, the differentiability index is 1.

(d) When $a = 0$ the system becomes

$$\dot{z}_1 = 0$$
$$z_1 = u$$

which is not solvable as soon as u is not a constant. Furthermore, z_2 is free, hence the solution is never unique.