Solution for the “Modeling and Simulation" exam (TSRT62) 2018-10-29

1. (a) Validating a model means testing it with new data. It allows to check the quality of a fitted model: if prediction errors are much bigger for the validation data than for the estimation data then the model is inadequate. In some cases, validation can give hints of what the problem could be (e.g. overfitting, wrong model class, etc.).

   (b) The closed loop system is

$$y(t) + (a + bk)y(t - 1) = e(t)$$

   Even if k is known, all pairs (a, b) such that $a + bk = \text{const}$ would lead to the same predictor, hence the system is not identifiable. If the feedback law is modified for instance to $u(t) = -ky(t - 1)$ (i.e., a delay is added) then the closed-loop system is

$$y(t) + ay(t - 1) + bky(t - 2) = e(t)$$

   which is instead identifiable.

   (c) Since in general $D(q) \neq R(q) \neq 1$, the class of black-box models that can give an unbiased predictor is the BJ class.

   (d) Both graphs have a causality conflict, see Fig. 1. In both cases it can be resolved by adding a R-type element.

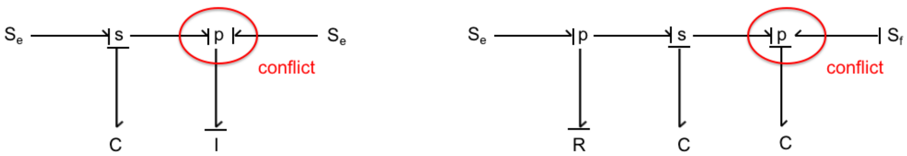

Figure 1: Exercise 1: causality conflict.

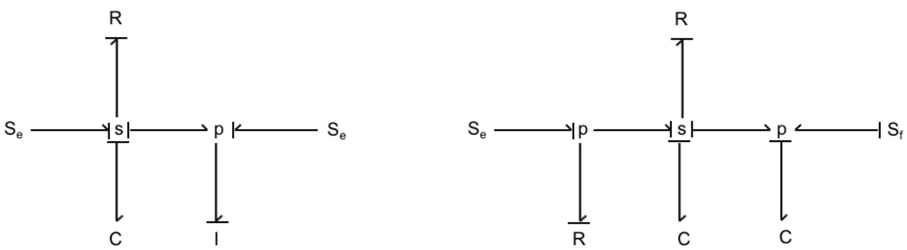

Figure 2: Exercise 1: causality conflict resolved

2. Exercise 2.

   (a) Denote $\theta = \begin{bmatrix} a_1 \\ b_1 \end{bmatrix}$. Since the true system lies in the chosen model class and the excitation is sufficiently rich (u is a white noise), it is $\hat{\theta}_N \to \theta^* = \theta_0 = \begin{bmatrix} 0.4 \\ 0.2 \end{bmatrix}$, i.e., the problem is unbiased and the model is identifiable. θ^* should be obtained by explicitly minimizing $\bar{V}(\theta) = \lim_{N\to\infty} V_N(\theta)$ (we do this in point (b) below). The formula for the variance of the estimates is $P_N \approx \frac{1}{N}\lambda_v \bar{R}^{-1}$ where $\bar{R} = E[\psi(t,\theta_0)\psi^T(t,\theta_0)]$ and $\psi(t,\theta) = \frac{d}{d\theta}\hat{y}(t|\theta)$ is the gradient of the predictor. In our case $\psi(t,\theta) = \begin{bmatrix} -y(t-1) \\ u(t-1) \end{bmatrix}$, which gives

$$\bar{R} = E\begin{bmatrix} y^2(t-1) & -y(t-1)u(t) \\ -y(t-1)u(t) & u^2(t) \end{bmatrix} = \begin{bmatrix} R_y(0) & -R_{yu}(-1) \\ -R_{yu}(-1) & R_u(0) \end{bmatrix}$$

Computing the terms:

$$\begin{aligned} R_y(0) &= E[y^2(t)] = E[(-0.4y(t-1) + 0.2u(t) + v(t))^2] \\ &= 0.16R_y(0) + 0.04\lambda_u + \lambda_v \\ &\quad - 0.16\underbrace{E[y(t-1)u(t)]}_{=R_{yu}(-1)=0} - 0.8\underbrace{E[y(t-1)v(t)]}_{=0} + 0.4\underbrace{E[u(t)v(t)]}_{=0} \end{aligned}$$

i.e., $R_y(0) = \frac{2.04}{0.84} = 2.42$. Hence

$$\bar{R} = \begin{bmatrix} 2.42 & 0 \\ 0 & 1 \end{bmatrix} \implies \bar{R}^{-1} = \begin{bmatrix} 0.41 & 0 \\ 0 & 1 \end{bmatrix}$$

and $P_N \approx \frac{2}{N}\bar{R}^{-1}$, meaning that $\text{Var}[\hat{a}_1] = \frac{0.82}{N}$ and $\text{Var}[\hat{b}_1] = \frac{2}{N}$.

   (b) Also in this case the true system is contained in the model class, and the excitation is rich, hence we have still an unbiased problem and an identifiable model. From this we already know that it must be $\hat{a}_1 = 0.4$, $\hat{a}_2 = 0$ and $\hat{b}_1 = 0.2$. Let us however compute explicitly these values through the prediction error minimization, as requested in the exercise.

$$\begin{aligned} \bar{V}(\theta) =& E[(y(t) - \hat{y}(t|\theta))^2] = E[((a_1 - 0.4)y(t-1) + a_2y(t-2) + (0.2 - b_1)u(t) + v(t))^2] \\ =& 2.42(a_1 - 0.4)^2 + 2.42a_2^2 + (0.2 - b_1)^2 + 2 - 1.94(a_1 - 0.4)a_2 \end{aligned}$$

since $R_y(0) = 2.42$ (same as before) and

$$R_y(1) = E[y(t)y(t-1)] = E[(-0.4y(t-1)+0.2u(t)+v(t))y(t-1)] = -0.4R_y(0)+0 = -0.97$$

Differentiating w.r.t. the parameters:

$$\begin{aligned} \frac{d\bar{V}(\theta)}{da_1} &= 4.84(a_1 - 0.4) - 1.94a_2 = 0 \\ \frac{d\bar{V}(\theta)}{da_2} &= -1.94(a_1 - 0.4) + 4.84a_2 = 0 \\ \frac{d\bar{V}(\theta)}{db_1} &= -0.4 + 2b_1 = 0 \end{aligned}$$

from which it is straightforward to see that $\hat{a}_1 = 0.4$, $\hat{a}_2 = 0$ and $\hat{b}_1 = 0.2$ is indeed a solution, in correspondence of which

$$\frac{d^2\bar{V}(\theta)}{da_1^2} > 0, \quad \frac{d^2\bar{V}(\theta)}{da_2^2} > 0, \quad \frac{d^2\bar{V}(\theta)}{db_1^2} > 0$$

i.e., a minimum of $\bar{V}(\theta)$. Let us compute the variance of these estimates.

$$\begin{aligned}\bar{R} =& E[\psi(t,\theta_0)\psi^T(t,\theta_0)] = E\begin{bmatrix}-y(t-1)\\ -y(t-2)\\ u(t)\end{bmatrix}\begin{bmatrix}-y(t-1) & -y(t-2) & u(t)\end{bmatrix}\\ =& \begin{bmatrix}R_y(0) & R_y(1) & -R_{yu}(-1)\\ R_y(1) & R_y(0) & -R_{yu}(-2)\\ -R_{yu}(-1) & -R_{yu}(-2) & R_u(0)\end{bmatrix}\end{aligned}$$

$R_y(1) = -0.97$ (already computed). $R_{yu}(-1) = E[y(t-1)u(t)] = 0$, $R_{yu}(-2) = 0$. Therefore

$$\bar{R} = \begin{bmatrix}2.42 & -0.97 & 0\\ -0.97 & 2.42 & 0\\ 0 & 0 & 1\end{bmatrix} \implies \bar{R}^{-1} = \begin{bmatrix}0.49 & 0.19 & 0\\ 0.19 & 0.49 & 0\\ 0 & 0 & 1\end{bmatrix}$$

or $\text{Var}[\hat{a}_1] = \text{Var}[\hat{a}_2] = \frac{0.98}{N}$, $\text{Var}[\hat{b}_1] = \frac{2}{N}$.

(c) For any $n_a > 1$ and $n_b > 1$, it will be $\hat{a}_1 = 0.4$, $\hat{a}_2 = \ldots = \hat{a}_{n_a} = 0$ and $\hat{b}_1 = 0.2$, $\hat{b}_2 = \ldots = \hat{b}_{n_b} = 0$ (model is still unbiased and identifiable), but the variance of the estimate will increase with n_a and n_b (compare the change in $\text{Var}[\hat{a}_1]$ in (a) and (b) above).

3. Exercise 3: System identification.

(a) The AXR order selection tool suggests a delay $n_k = 1$. Keeping this delay, a model that satisfies the constraint of max 8 parameters is for instance the following OE(3,3,1):

```
oe331 =
Discrete-time OE model:  y(t) = [B(z)/F(z)]u(t) + e(t)

 B(z) = 0.3867 (+/- 0.001029) z^-1 + 0.09289 (+/- 0.00337) z^-2
 - 0.195 (+/- 0.0009081) z^-3


  F(z) = 1 - 1.398 (+/- 0.004774) z^-1 + 1.296 (+/- 0.004994) z^-2
  - 0.3309 (+/- 0.004437) z^-3

Name: oe331
```

```
Sample time: 0.5 seconds

Parameterization:
   Polynomial orders:   nb=3   nf=3   nk=1
   Number of free coefficients: 6
   Use "polydata", "getpvec", "getcov" for parameters and their uncertainties.

Status:
Termination condition: Near (local) minimum, (norm(g) < tol).
Number of iterations: 3, Number of function evaluations: 11

Estimated using PEM on time domain data "mydatade".
Fit to estimation data: 99.08\%
FPE: 0.0002071, MSE: 0.0001997
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It provides a fit to validation data of 99.05%.

Parameter uncertainty is small.

The model fit is shown in red in Fig. 3. Residuals are in Fig. 4 and are within

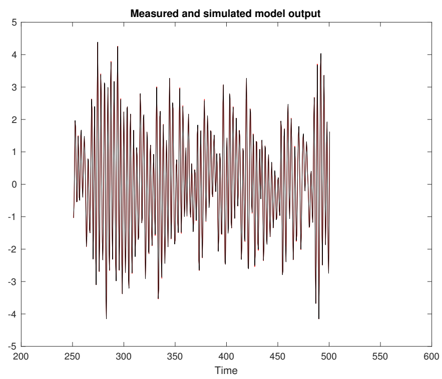

Figure 3: Model fit for OE(3,3,1).

ranges. Zeros/poles are in Fig. 5. The poles are all stable. Confidence interval are very small. The Bode plot is in Fig. 6.

(b) The Bode plot shows a single resonant peak at 2 rad/sec, hence the correct guess is (c). This is confirmed by a frequency function computed with ETFE or SPA.

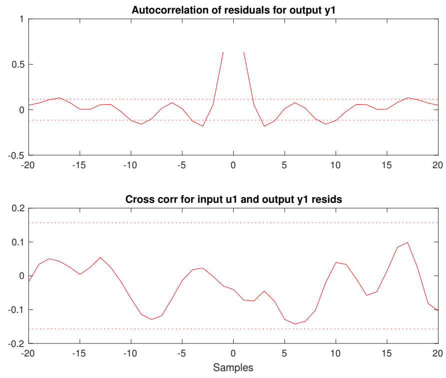

Figure 4: Residuals

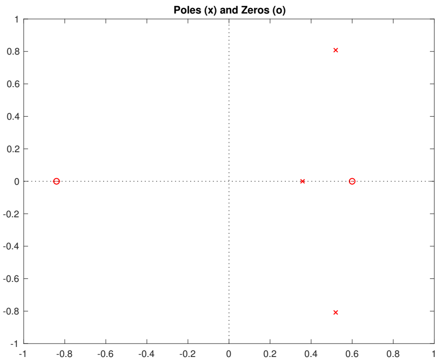

Figure 5: Zeros and poles

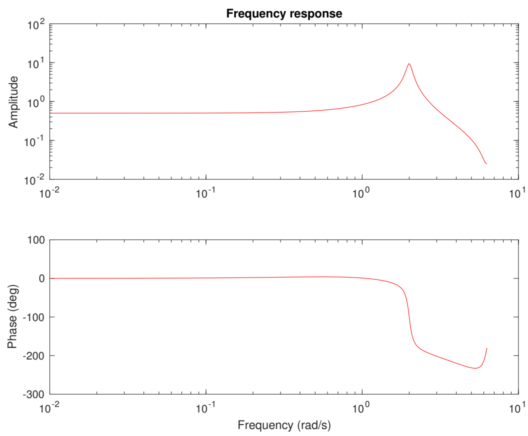

Figure 6: Bode plot of OE.

4. Exercise 4: bond graph

(a) The pressures $p_{in} = \rho g h_1$ and $p_2 = p_s + \rho g h_2$, where g = gravitational constant, give: $\Delta p = p_s + \rho g(h_2 - h_1)$. Of this pressure, a part is an external source, due to gravity: $p_g = \rho g(h_2 - h_1)$, while p_s is the pressure into an R-type element. The bond graph is given in Fig. 7 and its causality is conflict-free.

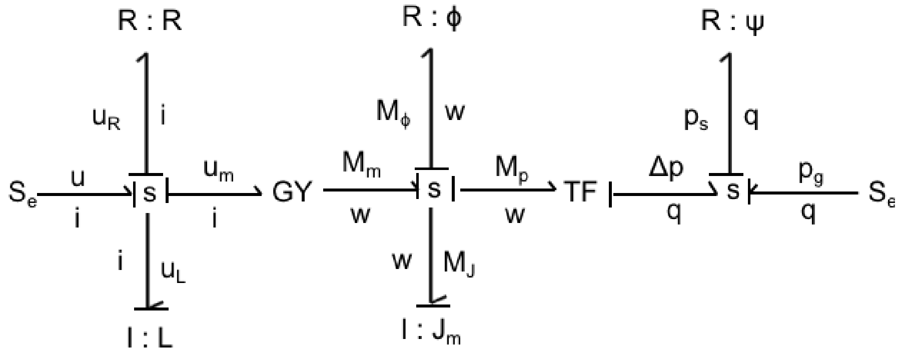

Figure 7: Exercise 4, bond graph

(b) The inputs are u and p_g, and the state variables are the flow variables at the I-elements: i and ω. With the choice of variable names shown in Fig. 7 one gets

$$
\begin{array}{rcl}
\text{s:} & u & = u_R + u_L + u_m \\
\text{R:} & u_R & = Ri \\
\text{I:} & L\frac{di}{dt} & = u_L \\
\text{GY:} & M_m & = \beta i, \qquad \omega = \dfrac{1}{\beta} u_m \\
\text{s:} & M_m & = M_\phi + M_p + M_J \\
\text{R:} & M_\phi & = \phi(\omega) \\
\text{I:} & J_m\frac{d\omega}{dt} & = M_J \\
\text{TF:} & q & = k_p\omega, \qquad \Delta p = \dfrac{1}{k_p} M_p \\
\text{s:} & \Delta p & = p_g + p_s \\
\text{R:} & p_s & = \psi(q)
\end{array}
$$

which yields the following system of ODEs

$$
\begin{aligned}
\frac{di}{dt} &= \frac{1}{L}\left(u - Ri - \beta\omega\right) \\
\frac{d\omega}{dt} &= \frac{1}{J_m}\left(\beta i - \phi(\omega) - k_p p_g - k_p \psi(k_p\omega)\right)
\end{aligned}
$$

(c) Since neither $\phi(\cdot)$ nor $\psi(\cdot)$ is inverted in the state space equation, there is no difference.

5. Exercise 5. Denote $z = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

(a) In matrix form $E\dot{z} + Fz = Gu$, the system is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{z} + \begin{bmatrix} 0 & 0 & 2 \\ -1 & 0 & -3 \\ 1 & 1 & 0 \end{bmatrix} z = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} u$$

where E is not invertible. Differentiating the last row

$$\tilde{E}\dot{z} + \tilde{F}z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \dot{z} + \begin{bmatrix} 0 & 0 & 2 \\ -1 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix} z = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \dot{u}$$

$\tilde{E}$ is still not invertible. Subtracting the first and second rows to the third one gives:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{z} + \begin{bmatrix} 0 & 0 & 2 \\ -1 & 0 & -3 \\ 1 & 0 & 1 \end{bmatrix} z = \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix} u + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \dot{u}$$

and, differentiating the last row again

$$\hat{E}\dot{z} + \hat{F}z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \dot{z} + \begin{bmatrix} 0 & 0 & 2 \\ -1 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix} z = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix} \dot{u} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \ddot{u}$$

Now $\hat{E}$ is invertible hence we can stop. The system has differentiability index =2.

(b) When $u = k_i x_i$, the closed loop system in matrix form has still E singular. In order for the new $\tilde{E}$ to have full rank, the only possibility is to use $u = k_3 x_3$. In fact in this case

$$E\dot{z} + Fz = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{z} + \begin{bmatrix} 0 & 0 & 2 \\ -1 & 0 & -(3+2k_3) \\ 1 & 1 & -k_3 \end{bmatrix} z = 0$$

and after differentiation of the third row

$$\tilde{E}\dot{z} + \tilde{F}z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & -k_3 \end{bmatrix} \dot{z} + \begin{bmatrix} 0 & 0 & 2 \\ -1 & 0 & -(3+2k_3) \\ 0 & 0 & 0 \end{bmatrix} z = 0$$

When $k_3 \neq 0$, $\tilde{E}$ is invertible, hence diff. index =1.

(c) Inserting the feedback law:

$$E\dot{z} + Fz = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{z} + \begin{bmatrix} 0 & 0 & 2 \\ -(1+2k_1) & -2k_2 & -(3+2k_3) \\ 1-k_1 & 1-k_2 & -k_3 \end{bmatrix} z = 0$$

In order for this system to be non-uniquely solvable it has to be $\det(sE+F) = 0$ $\forall\, s$. Computing:

$$\det(sE+F) = \det \begin{bmatrix} s & 0 & 2 \\ -(1+2k_1) & s-2k_2 & -(3+2k_3) \\ 1-k_1 & 1-k_2 & -k_3 \end{bmatrix}$$

$$= s\left(-(s-2k_2)k_3 + (1-k_2)(3+2k_3) - 2(1+2k_1)(1-k_2) - 2(1-k_1)(s-2k_2)\right) = 0 \;\; \forall\, s$$

i.e. $k_1 = 1$, $k_2 = 1$, and $k_3 = 0$. For all other values the system is uniquely solvable.