Solution for the "Modeling and Simulation" exam (TSRT62) 2018-08-27

1. (a) A step response can give information for instance on the time constants of the system, on the sampling time, on the presence of oscillations, on its linear/nonlinear behavior, etc.

   (b) The first order implicit Adams method is nothing else than the Euler backwards method. On the test function $\dot{x} = \lambda x$, it becomes

$$x_{n+1} = x_n + h\lambda x_{n+1}$$

   i.e.,

$$x_{n+1} = \frac{1}{1 - h\lambda} x_n$$

   The stability region is $|1 - h\lambda| > 1$, i.e., $h\lambda$ must be outside a disk of radius 1 centered at 1. When $\lambda < 0$ this is true for all $h > 0$. For us it is $\lambda = -3$, hence the Euler backwards discretization of our system $\dot{x} = -3x$ is stable for all $h > 0$.

   (c) Stiff systems are difficult because fast and slow modes coexist. One simple way to quantify stiffness of a (linear) system is to look at its eigenvalues. The eigenvalues of the 3 systems are $\{-1, -2\}$, $\{-0.001, -1\}$, and $\{-1, -100\}$. The ratio between them is 2, 1000 and 100. Hence (i) is the least stiff system, followed by (iii), while (ii) is the stiffest.

   (d) Modelica is an object-oriented language. When you connect blocks constructed in this language, some of the variables may become algebraic equations (describing the block interconnections), rather than differential equations.

2. Asymptotically it is

$$\begin{aligned}
V(b) &= \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} [\,\hat{y}(t) - y(t)\,]^2 \\
&= \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} [\,bu(t-1) - u(t-1) - u(t-2) - w(t)\,]^2 \\
&= (b-1)^2 R_u(0) + R_u(0) + R_w(0) - 2(b-1)R_u(1) \\
&- 2(b-1)R_{uw}(1) + 2R_{uw}(2),
\end{aligned}$$

   where

$$R_{ab}(\tau) = E[(a(t) - \bar{a})(b(t-\tau) - \bar{b})] = \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{\infty} (a(t) - \bar{a})(b(t-\tau) - \bar{b})$$

   Since w and u are independent, it is $R_{uw}(\tau) = 0$ for all τ, which gives

$$V(b) = (b-1)^2 R_u(0) + R_u(0) + \lambda_w - 2(b-1)R_u(1)$$

(a) If u is a white noise (of mean 0 and variance λ_u) it is

$$R_u(\tau) = \begin{cases} \lambda_u & \text{if } \tau = 0 \\ 0 & \text{if } \tau \neq 0 \end{cases}$$

hence

$$\min_b V(b) = \min_b [(b-1)^2 \lambda_u + \lambda_u + \lambda_w]$$

yields $b = 1$.

(b) $u(t) = (-1)^t$ gives $R_u(0) = 1$, $R_u(1) = -1$ etc. Hence

$$\min_b V(b) = \min_b [(b-1)^2 + 1 + \lambda_w + 2(b-1)] = \min_b [b^2 + \lambda_w]$$

gives $b = 0$.

(c) In the final case it is

$$\min_b V(b) = \min_b [(b-1)^2 + 1 + \lambda_w - 2(b-1)] = \min_b [(b-2)^2 + \lambda_w]$$

which is minimized for $b = 2$.

3. Exercise 3: System identification.

There is no sign of resonances, see the frequency function in Fig. 1, computed with SPA (M=100).

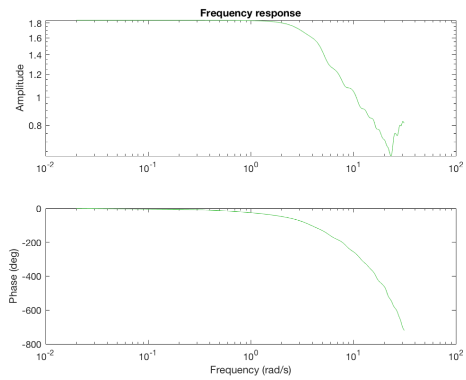

Figure 1: Frequency function.

The AXR order selection tool suggests a delay $n_k = 3$. Keeping this delay, a model that satisfies the constraint of max 3 poles is for instance the following OE(2,3,3):

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oe233 =
Discrete-time OE model:  y(t) = [B(z)/F(z)]u(t) + e(t)
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  B(z) = 0.008766 (+/- 0.008597) z^-3 + 0.9868 (+/- 0.01131) z^-4

  F(z) = 1 - 0.4045 (+/- 0.01094) z^-1 - 0.1036 (+/- 0.0141) z^-2
            - 0.001491 (+/- 0.008743) z^-3

Name: oe233
Sample time: 0.1 seconds

Parameterization:
   Polynomial orders:   nb=2   nf=3   nk=3
   Number of free coefficients: 5
   Use "polydata", "getpvec", "getcov" for parameters and their uncertainties.

Status:
Termination condition: Near (local) minimum, (norm(g) < tol).
Number of iterations: 2, Number of function evaluations: 5

Estimated using PEM on time domain data "mydatade".
Fit to estimation data: 93.77\%
FPE: 0.01221, MSE: 0.01197
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It provides a fit to validation data of 92.69%.

Parameter uncertainty is reasonably small, although for $B(z)$ the first coefficient is of the same magnitude of the error (other models with max 3 poles seems to have a similar problem).

The model fit is shown in cyan in Fig. 2. Residuals are in Fig. 3 and are within

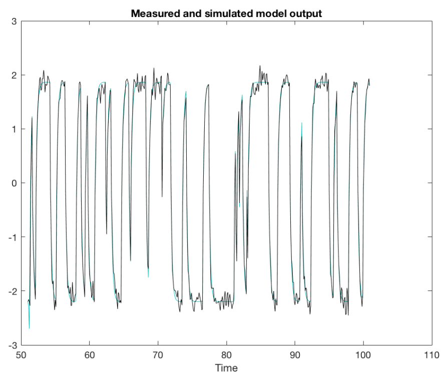

Figure 2: Model fit for OE(2,3,3).

ranges. Zeros/poles are in Fig. 5. The poles are all stable. Confidence interval are not completely disjoint, but again this appears due to the pole-order constraint. The frequency function is similar to the SPA frequency function, see Fig. 5, up to a difference in the DC gain.

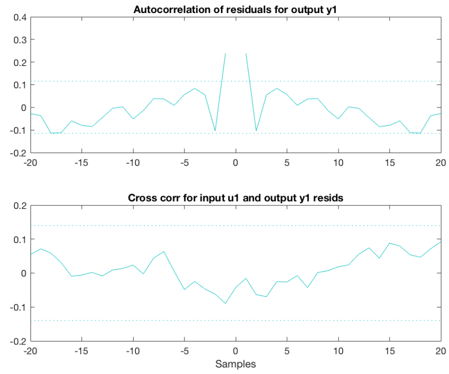

Figure 3: Residuals

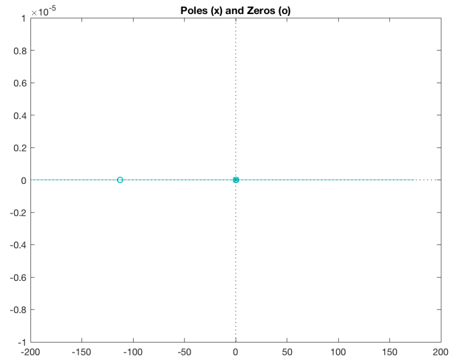

Figure 4: Zeros and poles

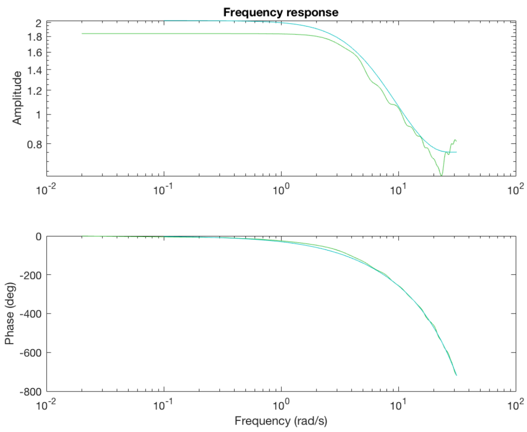

Figure 5: Frequency function of OE and SPA.

4. Exercise 4: bond graph

(a) Denote $(F_a,\, v_a)$ the (force, velocity) pair at the point a (top of the lever), $(F_j,\, v_j)$ those at the junction point between the lever and the piston shaft (point j in the figure), and $(F_b,\, v_b)$ the (force, velocity) pair at the point b. Comparing points a and j, the equation for the lever gives

$$\begin{aligned} \text{torque is the same:} \quad & F_a(L_1+L_2) &= F_j L_2 \\ \text{angular velocity is the same:} \quad & \tfrac{v_a}{L_1+L_2} &= \frac{v_j}{L_2} \end{aligned}$$

If $(p_j,\, Q_j)$ and $(p_b,\, Q_b)$ are the (pressure, flow) pairs at the pistons j and b, then flow incompressibility and no loss of energy in the liquid imply $p_j = p_b$ and $Q_j = Q_b$. Using $F_j = A_1 p_j$, $F_b = A_2 p_b$ and the equations above for the lever, one gets

$$F_b = A_2 p_b = A_2 p_j = \frac{A_2}{A_1} F_j = \frac{A_2}{A_1}\frac{L_1+L_2}{L_2} F_a$$

and similarly

$$v_b = \frac{A_1}{A_2}\frac{L_2}{L_1+L_2} v_a$$

Hence $n = \frac{A_2}{A_1}\frac{L_1+L_2}{L_2}$ is the transformation factor of a single TF mapping from $(F_a,\, v_a)$ to $(F_b,\, v_b)$.

(b) The bond graph is given in Fig. 6 and its causality is conflict-free.

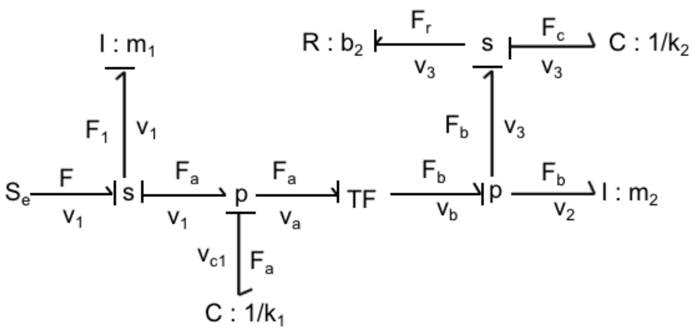

Figure 6: Exercise 4, bond graph

(c) Let us choose F as input and as state variables the effort variables at the C-elements and the flow variables at the I-elements:

$$x = \begin{pmatrix} v_1 & F_a & F_c & v_2 \end{pmatrix}^T$$

With the choice of variable names shown in Fig. 6 one gets

$$
\begin{array}{rll}
\text{s:} & F = & = F_1 + F_a \\
\text{I:} & m_1 \dot{v}_1 & = F_1 \\
\text{p:} & v_1 & = v_a + v_{c1} \\
\text{C:} & 1/k_1 \dot{F}_a & = v_{c1} \\
\text{TF:} & F_b & = n F_a \qquad v_b = \frac{1}{n} v_a \\
\text{p:} & v_b & = v_2 + v_3 \\
\text{s:} & F_b & = F_r + F_c \\
\text{I:} & m_2 \dot{v}_2 & = F_b \\
\text{C:} & 1/k_2 \dot{F}_c & = v_3 \\
\text{R:} & F_r & = b_2 v_3
\end{array}
$$

which yields the following ODEs

$$
\begin{aligned}
\dot{v}_1 &= \frac{1}{m_1}(F - F_a) \\
\dot{F}_a &= k_1 \left(v_1 - n v_2 - \frac{n^2}{b_2} F_a + \frac{n}{b_2} F_c \right) \\
\dot{F}_c &= \frac{k_2}{b_2}(n F_a - F_c) \\
\dot{v}_2 &= \frac{n}{m_2} F_a
\end{aligned}
$$

5. Exercise 5.

   (a) The bond graph is in Fig 7 and it has conflict-free causality.

   (b) Calling v the voltage at the generator and $u_R \,/\, u_L \,/\, u_C$ and $i_R \,/\, i_L \,/\, i_C$ the voltages and currents at R, L and C ($u_C = u_L$), the state variables are u_C and i_L, and

$$
\begin{array}{rll}
\text{s:} & v & = u_R + u_C \\
\text{R:} & u_R & = R i_R \\
\text{p:} & i_R & = i_L + i_C \\
\text{C:} & C \frac{du_C}{dt} & = i_C \\
\text{I:} & L \frac{di_L}{dt} & = u_C
\end{array}
$$

which gives

$$
\frac{d}{dt} \begin{bmatrix} u_C \\ i_L \end{bmatrix} = \begin{bmatrix} \frac{-1}{RC} & \frac{-1}{C} \\ \frac{1}{L} & 0 \end{bmatrix} \begin{bmatrix} u_C \\ i_L \end{bmatrix} + \begin{bmatrix} \frac{1}{RC} \\ 0 \end{bmatrix} v
$$

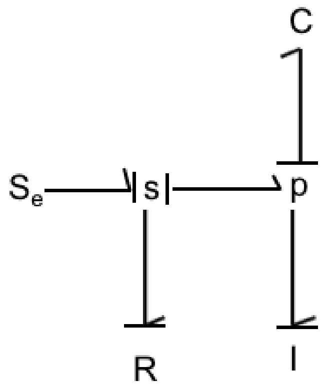

Figure 7: Exercise 5, bond graph for circuit with voltage generator

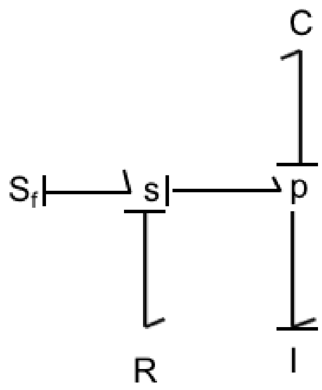

Figure 8: Exercise 5, bond graph for circuit with current generator

(c) With a current generator, the new bond graph is in Fig 8 and it also has conflict-free causality. Notice the change in direction of the R-block causality stroke.

(d) With the nonlinear model, for the first circuit the nonlinear ODEs are

$$\frac{du_C}{dt} = \frac{1}{C_1} i_C + \frac{1}{C_2} i_C^3 \quad (1)$$

$$\frac{di_L}{dt} = \frac{1}{L_1} u_C + \frac{1}{L_2} u_C^5 \quad (2)$$

The right hand side contains i_C which is not one of the two state variables, hence we need to add another equation to (1)-(2). In fact, to get rid of it, one must use $i_C = i_R - i_L$ to express i_C as a function of u_C, i_L and of the input v. i_R itself is not a state variable, therefore it must also be eliminated. However, $u_R = v - u_C = R_1 i_R + R_2 i_R^7$ cannot be used to expressed i_R (and hence i_C) in terms of u_C and v, hence one needs to differentiate this expression and pair it with the two ODEs above.

$$\frac{du_R}{dt} = \frac{dv}{dt} - \frac{du_C}{dt} = \left(R_1 + 7R_2 i_R^6\right) \frac{di_R}{dt}$$

or, from $i_C = i_R - i_L$,

$$\frac{di_C}{dt} = \frac{1}{(R_1 + 7R_2(i_C + i_L)^6)} \left(\dot{v} - \frac{1}{C_1} i_C + \frac{1}{C_2} i_C^3\right) - \frac{1}{L_1} u_C + \frac{1}{L_2} u_C^5 \qquad (3)$$

Notice that $\frac{1}{(R_1+7R_2(i_C+i_L)^6)}$ is always well-posed when $R_1 \neq 0$ (i.e., the "Jacobian" g_y in eq. (7.31) of the book is always invertible). Now (1)-(3) is a system of ODEs in u_C, i_L, i_C and input v, $\dot{v}$. We conclude that the differentiability index is 1.

If instead we consider the second circuit, the current i_R is now also the input of the system. Hence in (1)-(2) we can directly replace i_C with $i_R - i_L$ (an input and a state variable). The differentiability index is 0 in this case.