Solution for the "Modeling and Simulation" exam (TSRT62) 2018-01-05

1. (a) A stiff system has time constants that are very different, which typically make simulations slow because to avoid instability a short step size must be chosen.

   (b) Identifying the parameters of an ARX model is easier and quicker because the model is linear in the parameters, hence least square methods can be used and provide a global optimum. This is not true for OE models.

   (c) The eigenvalues of the system are $\lambda_1 = -1$ and $\lambda_2 = -4096$. Hence this is a stiff system, of which we are interested only in the fast part. Since $x(0)$ is the origin, a simplified version approximating the original one is:

$$\begin{aligned}\dot{x}_1 &= 0\\ \dot{x}_2 &= -4096x_2 + 512u\end{aligned}$$

   (d) On the test function $\dot{x} = \lambda x$, the Euler backwards method becomes

$$x_n = x_{n-1} + h\lambda x_n$$

   i.e.,

$$x_n = \frac{1}{1-h\lambda}x_{n-1}$$

   The stability region is $|1 - h\lambda| > 1$, i.e., $h\lambda$ must be outside a disk of radius 1 centered at 1. When $\lambda < 0$ this is always true, hence the Euler backwards discretization of the stable system $\dot{x} = -10x$ is stable for all $h > 0$. Instead $\dot{x} = 10x$ is unstable, hence if we want also its Euler discretization to be unstable it must be $0 < h < 0.2$.

2. Exercise 2

   (a) One can compute the prediction error minimization through

$$\begin{aligned}\bar{V}(\theta) &= \lim_{N\to\infty}\frac{1}{N}\sum_{t=1}^{N}\left(y(t) - \hat{y}(t|\theta)\right)^2\\ &= \mathbb{E}\left(0.5u(t-1) + 0.3u(t-2) + w(t) - b_1u(t-1) - b_2u(t-2)\right)^2\\ &= \left((0.5-b_1)^2 + (0.3-b_2)^2\right)R_u(0) + \lambda_w\\ &\quad + 2(0.5-b_1)(0.3-b_2)\underbrace{R_u(1)}_{=0} + 2(0.5-b_1)\underbrace{R_{uw}(1)}_{=0} + 2(0.3-b_2)\underbrace{R_{uw}(2)}_{=0}\end{aligned}$$

   which is minimized by $\hat{b}_1 = 0.5$ and $\hat{b}_2 = 0.3$. Since the estimation is unbiased, we can use the following formula for the covariance of the parameters

$$P_N \approx \frac{\lambda_w}{N}\bar{R}^{-1}$$

where

$$\bar{R} = \mathbb{E}(\psi(t,\theta)\psi(t,\theta)^T), \qquad \psi(t,\theta) = \frac{d}{d\theta}\hat{y}(t|\theta)$$

Computing:

$$\psi(t,\theta) = \begin{bmatrix} u(t-1) \\ u(t-2) \end{bmatrix} \implies \bar{R} = \begin{bmatrix} R_u(0) & R_u(1) \\ R_u(1) & R_u(0) \end{bmatrix} = \begin{bmatrix} \lambda_u & 0 \\ 0 & \lambda_u \end{bmatrix}$$

The variance of the parameters is therefore given by the diagonal terms of P_N:

$$\mathrm{Var}(b_i) = \frac{\lambda_w}{N\lambda_u} = \frac{2}{3N}$$

(b) In this case for $N \to \infty$ we have

$$\begin{aligned}
\bar{V}(\theta) &= \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} (y(t) - \hat{y}(t|\theta))^2 \\
&= \mathbb{E}\left((0.5 - b_2)u(t-1) + (0.3 - b_2)u(t-2) + w(t)\right)^2 \\
&= \Big((0.5 - b_1)^2 + (0.3 - b_2)^2\Big) \underbrace{R_u(0)}_{=1} + \lambda_w + 2(0.5 - b_1)(0.3 - b_2) \underbrace{R_u(1)}_{=1} \\
&\quad + 2(0.5 - b_1) \underbrace{R_{uw}(1)}_{=0} + 2(0.3 - b_2) \underbrace{R_{uw}(2)}_{=0} \\
&= \Big((0.5 - b_1 + 0.3 - b_2)^2\Big) + \lambda_w
\end{aligned}$$

which is minimized for

$$b_1 + b_2 = 0.8$$

Since there is an infinite number of pairs $(b_1,\, b_2)$ solving this equation, the system is not identifiable.

(c) Observe that we have the right class of models for the true system, hence the problem is unbiased. Using eq. 12.46 (Swedish book, eq. 11.60 in English book),

$$\theta_* = \lim_{N\to\infty} \hat{\theta}_N = \arg\min_\theta \int_{-\pi}^{\pi} \left|G_0(e^{i\omega}) - G(e^{i\omega},\theta)\right|^2 \frac{\Phi_u(\omega)}{|H_*(e^{i\omega})|^2} d\omega$$

where $G_0(e^{i\omega})$ is the true system, $G(e^{i\omega},\theta)$ is the model, $H_*(e^{i\omega}) = 1$ and $\Phi_u(\omega) = 1$. Since the problem is unbiased, there exists a value θ_o such that $G_0(e^{i\omega}) = G(e^{i\omega},\theta_o)$, and it follows that $\theta_* = \theta_o$. In our case, $\hat{b}_1 = 0.5$, $\hat{b}_2 = 0.3$, $\hat{f}_1 = 0.2$ and $\hat{f}_2 = 0.4$.

(d) Since a constant signal is used as input, it means that the parameter fitting is concentrated exclusively at the $\omega = 0$ frequency, as $\Phi_u(\omega) = 0$ for $\omega \neq 0$. This gives

$$\hat{b} = \mathrm{argmin}_b |G(e^{i0}) - b|^2 \implies \hat{b} = \frac{0.5 + 0.3}{1 + 0.2 + 0.4} = \frac{1}{2}$$

3. Exercise 3: System identification.

   The ARX order selection tool of the System Id toolbox suggests ARX(9,10,2) which gives a fit of 63.07% to validation data, but this model has problems: too many zero-pole near-cancellations (see Fig. 3) and a noise-output estimated spectrum which is incompatible with the one suggested in the text (see Fig. 5). Hence it is not an acceptable model. Notice that to obtain the desired noise-output TF, OE cannot be used. It can also be shown that for basically all choices of model order also ARMAX models do not lead to good noise-output estimated transfer function. Hence BJ models are the best candidates. For instance a BJ(3,2,4,4,1) yields a fit 65.1% to validation data.

```
bj32441 =
Discrete-time BJ model:  y(t) = [B(z)/F(z)]u(t) + [C(z)/D(z)]e(t)
  B(z) = -0.01016 (+/- 0.008964) z^-1 + 1.013 (+/- 0.02996) z^-2
             - 0.8333 (+/- 2.637) z^-3

  C(z) = 1 - 0.06174 (+/- 0.3814) z^-1 - 0.6966 (+/- 0.3184) z^-2


  D(z) = 1 - 1.082 (+/- 0.3845) z^-1 - 0.7019 (+/- 0.6886) z^-2
             + 0.7804 (+/- 0.2688) z^-3 + 0.02457 (+/- 0.07272) z^-4


  F(z) = 1 - 1.717 (+/- 2.626) z^-1 + 1.058 (+/- 2.333) z^-2
            - 0.3427 (+/- 0.8533) z^-3 + 0.06219 (+/- 0.203) z^-4

Name: bj32441
Sample time: 0.1 seconds

Parameterization:
   Polynomial orders:   nb=3   nc=2   nd=4   nf=4   nk=1
   Number of free coefficients: 13

Estimated using PEM on time domain data "mydatade".
Fit to estimation data: 93.14% (prediction focus)
FPE: 0.02335, MSE: 0.02216
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   Parameter uncertainty is very small for $B(z)$ and $C(z)$, but it is big for $D(z)$ and $F(z)$, i.e., the noise-output transfer function is identified much more precisely than the input-output transfer function. This is apparently unavoidable in this case.

   The model fit is shown in orange in Fig. 1. Residuals are in Fig. 2 and are within ranges for both models. Zeros/poles are in Fig. 3. The poles are all stable. For BJ, the confidence intervals are very small. For ARX instead, most zero/poles pairs are overlapping. The estimated Bode plots for the two models are similar, and similar to the SPA frequency function, see Fig. 4. The resulting noise spectrum, shown in Fig. 5, is however very different: only that of BJ resembles the noise-output transfer function given in the text of the exercise.

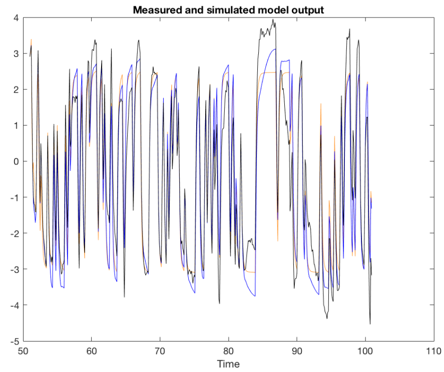

Figure 1: Model fit for ARX(9,10,2) (blue) and for BJ(3,2,4,4,1) (orange).

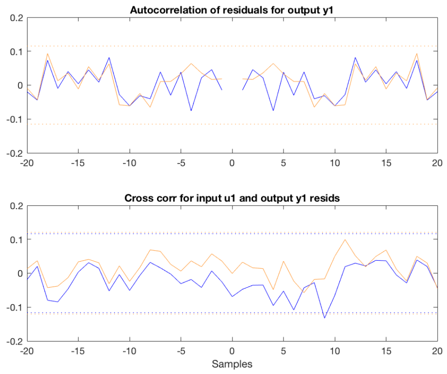

Figure 2: Residuals

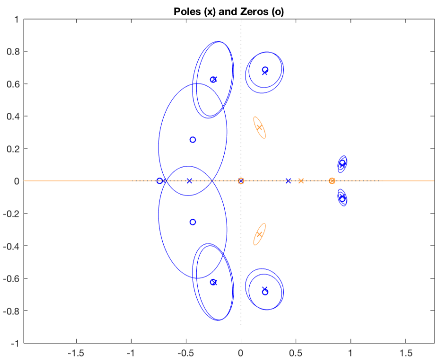

Figure 3: Zeros and poles

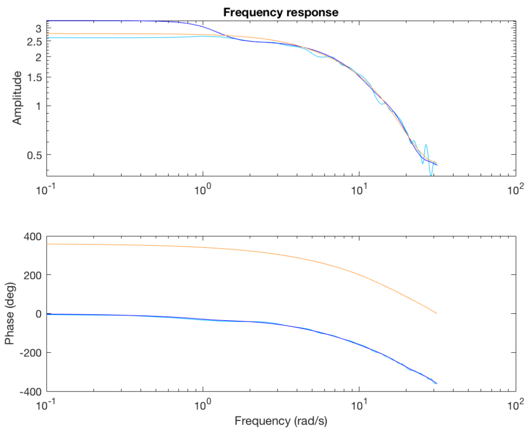

Figure 4: Bode plot of the estimated ARX and BJ input-output transfer functions, and frequency function computed with SPA (M=50) (cyan).

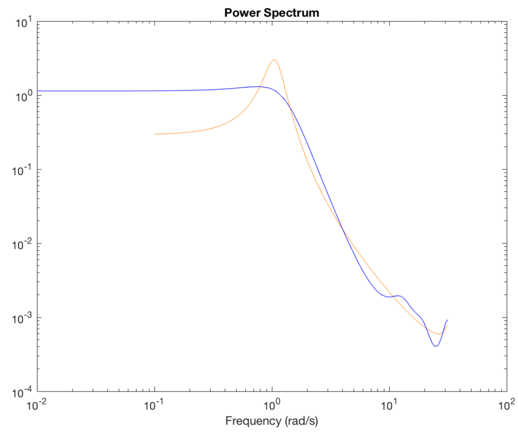

Figure 5: Bode plot of the estimated ARX and BJ noise-output transfer functions.

4. Exercise 4: bond graph

   (a) The bond graph is given in Fig. 6 and its causality is conflict-free.

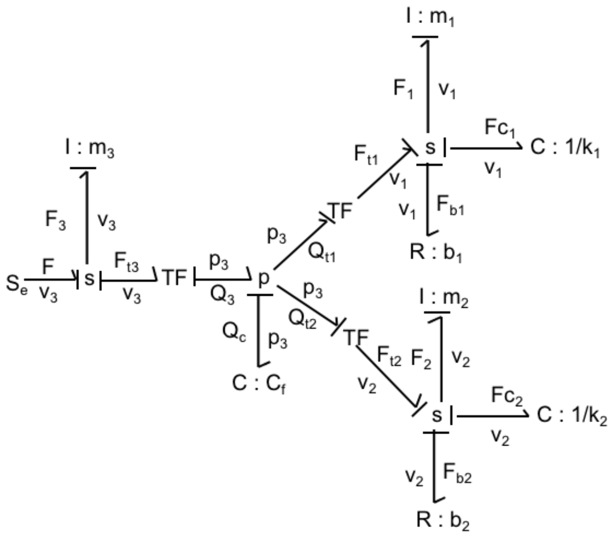

Figure 6: Exercise 4, bond graph

   (b) Let us choose F as input and as state variables the effort variables at the C-elements and the flow variables at the I-elements:

$$x = \begin{pmatrix} v_3 & p_3 & v_1 & v_2 & F_{c1} & F_{c2} \end{pmatrix}^T$$

   Denote $C_f = \frac{A_4}{\rho g}$ the fluid capacitance. With the choice of variable names shown in Fig. 6 one gets

$$
\begin{array}{rrll}
\text{s:} & F = & = F_3 + F_{t3} & \\
\text{I:} & m_3\dot{v}_3 & = F_3 & \\
\text{TF:} & F_{t3} & = A_3 p_3 & v_3 = Q_3/A_3 \\
\text{p:} & Q_3 & = Q_c + Q_{t1} + Q_{t2} & \\
\text{C:} & C_f \dot{p}_3 & = Q_c & \\
\text{TF:} & F_{t1} & = A_1 p_3 & v_1 = Q_{t1}/A_1 \\
\text{TF:} & F_{t2} & = A_2 p_3 & v_2 = Q_{t2}/A_2 \\
\text{s:} & F_{t1} & = F_1 + F_{c1} + F_{b1} & \\
\text{s:} & F_{t2} & = F_2 + F_{c2} + F_{b2} & \\
\text{I:} & m_1\dot{v}_1 & = F_1 & \\
\text{I:} & m_2\dot{v}_2 & = F_2 & \\
\text{C:} & 1/k_1 \dot{F}_{c1} & = v_1 & \\
\text{C:} & 1/k_2 \dot{F}_{c2} & = v_2 & \\
\text{R:} & F_{b1} & = b_1 v_1 & \\
\text{R:} & F_{b2} & = b_2 v_2 &
\end{array}
$$

which yields the following ODEs

$$
\begin{aligned}
\dot{v}_3 &= \frac{1}{m_3}\left(F - A_3 p_3\right) \\
\dot{p}_3 &= \frac{1}{C_f}\left(A_3 v_3 - A_1 v_1 - A_2 v_2\right) \\
\dot{v}_1 &= \frac{1}{m_1}\left(A_1 p_3 - F_{c1} - b_1 v_1\right) \\
\dot{v}_2 &= \frac{1}{m_2}\left(A_2 p_3 - F_{c2} - b_2 v_2\right) \\
\dot{F}_{c1} &= k_1 v_1 \\
\dot{F}_{c2} &= k_2 v_2
\end{aligned}
$$

Expressing the pressure p_3 (a state variable) as $p_3 = \rho g h$, the output equation is

$$
y = h = \frac{p_3}{\rho g}
$$

5. Exercise 5: DAE.

(a) The bond graph is in Fig 7 and it has a causality conflict.

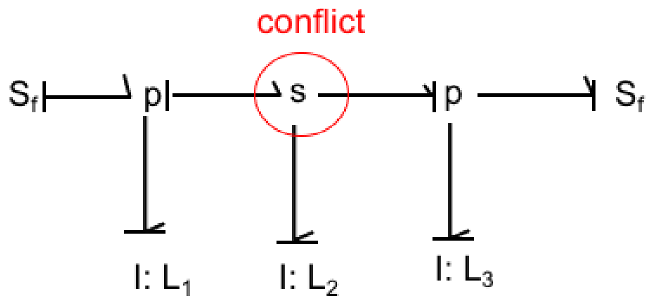

Figure 7: Exercise 5, bond graph

(b) Calling u_{L_j} the voltages at L_j, it is

$$u_{L_1} = u_{L_2} + u_{L_3} \qquad \text{plus} \qquad L_j \frac{di_j}{dt} = u_{L_j}, \quad j = 1,\ 2,\ 3$$

Denoting i_{S_j} the currents of the two generators, it is

$$\begin{aligned} i_{S_1} &= i_1 + i_2 \\ i_2 &= i_{S_2} + i_3 \end{aligned}$$

Choose as states $x = [i_1\ i_2\ i_3\ u_{L_2}\ u_{L_3}]$ and as input $w = [i_{S_1}\ i_{S_2}]$. The DAE system is then

$$\begin{bmatrix} L_1 & 0 & 0 & 0 & 0 \\ 0 & L_2 & 0 & 0 & 0 \\ 0 & 0 & L_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \dot{x} + \begin{bmatrix} 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \end{bmatrix} x = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} w$$

Differentiate the last two equations and regroup

$$\begin{bmatrix} L_1 & 0 & 0 & 0 & 0 \\ 0 & L_2 & 0 & 0 & 0 \\ 0 & 0 & L_3 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \end{bmatrix} \dot{x} + \begin{bmatrix} 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} x = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} w + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \dot{w}$$

The rank of the first matrix is still 3, hence we must differentiate again. Add 1st row (multiplied by $-1/L_1$) to the 4th, 2nd row (multiplied by $-1/L_2$) to the 4th and 5th rows, and 3rd row (multiplied by $1/L_3$) to the 5th:

$$\begin{bmatrix} L_1 & 0 & 0 & 0 & 0 \\ 0 & L_2 & 0 & 0 & 0 \\ 0 & 0 & L_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \dot{x} + \begin{bmatrix} 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & \frac{1}{L_1}+\frac{1}{L_2} & \frac{1}{L_1} \\ 0 & 0 & 0 & \frac{1}{L_2} & -\frac{1}{L_3} \end{bmatrix} x = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} w + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \dot{w}$$

and differentiate again the last two rows (and regroup):

$$\begin{bmatrix} L_1 & 0 & 0 & 0 & 0 \\ 0 & L_2 & 0 & 0 & 0 \\ 0 & 0 & L_3 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{L_1}+\frac{1}{L_2} & \frac{1}{L_1} \\ 0 & 0 & 0 & \frac{1}{L_2} & -\frac{1}{L_3} \end{bmatrix} \dot{x} + \begin{bmatrix} 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} x = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} (w+\dot{w}) + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \ddot{w}$$

It can be shown that now the matrix in front of $\dot{x}$ has rank 5 for all $L_j \neq 0$. Hence the differentiability index is 2.

(c) Replacing a current source with a voltage source, for instance as in Fig. 8, the bond graph still has a causality conflict.

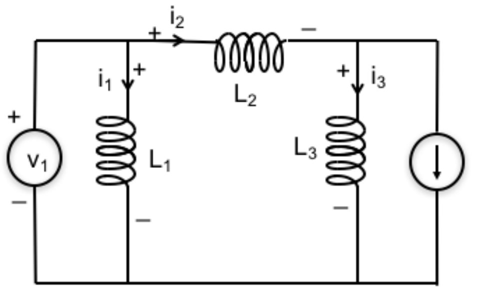

Figure 8: Exercise 5 (c)

The algebraic equations are

$$u_{L_1} = u_{L_2} + u_{L_3} = v_1$$
$$i_2 = i_{S_2} + i_3$$

Taking the same state vector as before and input $w = [v_1 \; i_{S_2}]$, the DAE is

$$\begin{bmatrix} L_1 & 0 & 0 & 0 & 0 \\ 0 & L_2 & 0 & 0 & 0 \\ 0 & 0 & L_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \dot{x} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} x = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} w$$

repeating the same procedure, it is easy to show that the differentiability index is still 2.