Solution for the "Modeling and Simulation" exam (TSRT62) 2017-10-24

1. (a) Letting $x_1 = y$ and $x_2 = \dot{y}$, then

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= (u - x_1)^{\frac{1}{3}} \\ y &= x_1 \end{cases}$$

(b) Rewrite $G(s)$ as

$$G(s) = \frac{\alpha/\beta}{1 + \beta^{-1}s}$$

then the time constant ($= \beta^{-1}$) and the static gain ($= \alpha/\beta$) can be found from a step response as in the plot in Fig. 1. From them α and β can be obtained.

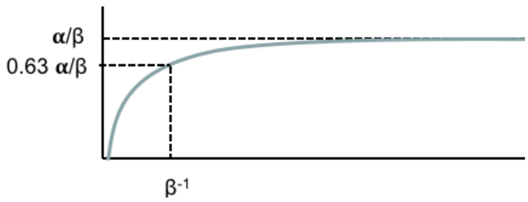

Figure 1: Step response of Ex. 1(b).

(c) The bond graph for the circuit is shown in Fig. 2. The conflict of causality is also shown. Adding a resistor in parallel as in Fig. 3 left, one gets the conflict-free bond graph of Fig. 3, right.

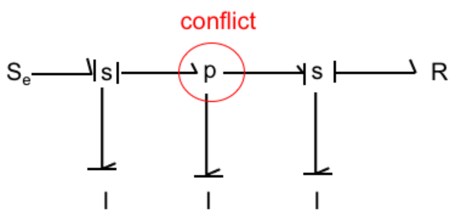

Figure 2: Bond graph for the circuit of Ex. 1(c).

(d) Using the formula

$$\Phi_y(\omega) = \left|G(e^{i\omega T})\right|^2 \Phi_u(\omega) = G(e^{i\omega T})G^*(e^{i\omega T})\Phi_u(\omega)$$

with $\Phi_u(\omega) = 1$, one gets

$$\Phi_y(\omega) = \frac{1.04 + 0.4\cos(\omega T)}{2 + 2\cos(\omega T)}$$

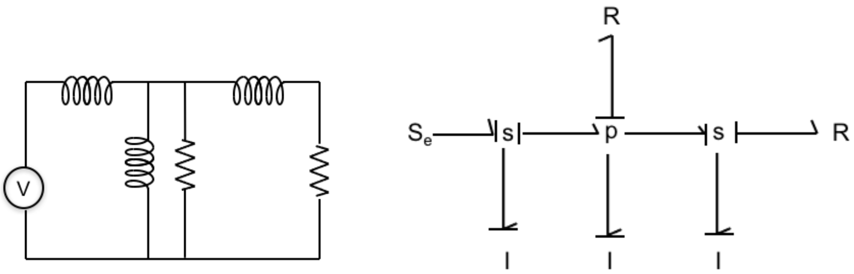

Figure 3: Circuit of Ex. 1(c) with extra resistor, and its bond graph.

2. Exercise 2

   (a) Of the 3 models proposed, only OE has the possibility of being unbiased, i.e., it "contains" the true system for a choice of the parameters ($b_1 = 1$, $b_2 = 2$ and $f_3 = 0$). Therefore OE is the most suitable model.

   (b) OE yields an unbiased estimation problem. FIR and ARX are instead biased in this case.

   (c) In the notation of the book, the OE predictor is

$$\begin{aligned}\hat{y}(t|\theta) &= \frac{B(q)}{F(q)}u(t) = \frac{b_1 + b_2 q^{-1}}{1 + f_3 q^{-3}}u(t) \\ &= -f_3\hat{y}(t-3|\theta) + b_1 u(t) + b_2 u(t-1)\end{aligned}$$

   with $\theta = (b_1,\, b_2,\, f_3)$, and the prediction error is

$$\epsilon(t,\theta) = y(t) - \hat{y}(t|\theta) = f_3\hat{y}(t-1|\theta) + (1-b_1)u(t) + (2-b_2)u(t-1) + e(t)$$

   When $N \to \infty$ we have

$$\begin{aligned}\mathbb{E}[\epsilon(t,\theta)^2] = \lim_{N\to\infty} V_N(\theta) &= f_3^2 R_{\hat{y}}(0) + (1-b_1)^2 R_u(0) + (2-b_2)^2 R_u(0) + \lambda_e \\ &+ f_3(1-b_1)\underbrace{\mathbb{E}[\hat{y}(t-3|\theta)u(t)]}_{=0} + f_3(2-b_2)\underbrace{\mathbb{E}[\hat{y}(t-3|\theta)u(t-1)]}_{=0} \\ &+ f_3\underbrace{\mathbb{E}[\hat{y}(t-3|\theta)e(t)]}_{=0} + (1-b_1)(2-b_2)\underbrace{R_u(1)}_{=0} \\ &+ (1-b_1)\underbrace{R_{ue}(0)}_{=0} + (2-b_2)\underbrace{R_{ue}(1)}_{=0}\end{aligned}$$

   Since $R_{\hat{y}}(0) \geq 0$ and $R_u(0) = \lambda_u > 0$, $\min_\theta \mathbb{E}[\epsilon(t,\theta)^2]$ occurs for $b_1^* = 1$, $b_2^* = 2$ and $f_3^* = 0$.

   (d) A model is identifiable if $\hat{y}(t|\theta^*) = \hat{y}(t|\theta)$ implies $\theta = \theta^*$. Here $\theta^* = (b_1^*\ b_2^*\ f_3^*) = (1\ 2\ 0)$, and corresponds to

$$\hat{y}(t|\theta) = u(t) + u(t-1)$$

while , $\theta = (b_1 \; b_2 \; f_3)$ is any other triple of parameter values. Clearly $\theta \neq \theta^*$ implies the predictor is different, since θ^* (and only θ^*) corresponds to the "exact" values. More formally, one can for instance use eq. 12.46 (Swedish book, eq. 11.60 in English book), with $G_0(e^{i\omega}) = 1 + 2e^{-i\omega}$ and $G(e^{i\omega}) = \frac{b_1+b_2e^{-i\omega}}{1+f_3e^{-3i\omega}}$, where we are assuming for simplicity that the sample time is $T = 1$. Since $\Phi_u(\omega) = \lambda_u = \text{const}$ and $H_*(e^{i\omega}) = 1$, it is

$$\int_{-\pi}^{\pi} \left| G_0(e^{i\omega}) - G(e^{i\omega}, \theta) \right|^2 \frac{\Phi_u(\omega)}{|H_*(e^{i\omega})|^2} d\omega = \int_{-\pi}^{\pi} \left| G_0(e^{i\omega}) - G(e^{i\omega}, \theta) \right|^2 d\omega$$

$$= \int_{-\pi}^{\pi} \left| 1 + 2e^{-i\omega} - \frac{b_1 + b_2 e^{-i\omega}}{1 + f_3 e^{-3i\omega}} \right|^2 d\omega$$

$$= \int_{-\pi}^{\pi} \left| \frac{(1 - b_1) + (2 - b_2)e^{-i\omega} + f_3 e^{-3i\omega} + 2 f_3 e^{-4i\omega}}{1 + f_3 e^{-3i\omega}} \right|^2 d\omega$$

which is clearly > 0 as soon as $\theta \neq \theta^*$. Hence identifiability follows.

3. Exercise 3: System identification.

   (a) Comparing for instance EFTE with $M = 100$ or with $M = 1000$ or SPA with $M = 1000$ yields the frequency functions of Fig. 4 which unanimously indicate

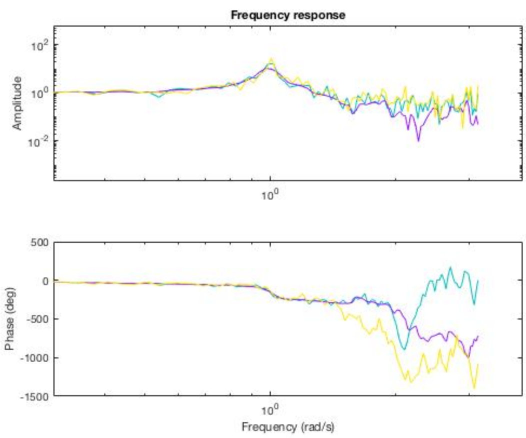

Figure 4: Frequency function with EFTE and SPA.

   that the system has a resonance peak at $\omega_1 = 1\text{rad/s}$.

   (b) This peak is clearly not compatible with G_1, but it is with G_2. G_3 has a second peak at $\omega_2 = 1 + 2\pi = 7.28\,\text{rad/s}$ which means that if it is used to generate the data and the data are sampled with $T = 1\text{s}$, because of aliasing, these data will be seen as if they were at $\omega' = \omega_1 = 1\,\text{rad/s}$ i.e., they will also produce the single empirical resonant peak observed in Fig. 4, hence also G_3 is compatible with the computed frequency function.

(c) Since there is a resonant peak, the TF from u to y has to have at least 2 poles. Let us try an OE model with $n_f = 3$ and $n_b = 3$, plus $n_k = 1$ as delay. The fit to validation data is 94.89% so the model is near-perfect for these data.

```
oe331 =
Discrete-time OE model:  y(t) = [B(z)/F(z)]u(t) + e(t)

 B(z) = 0.1205 (+/- 0.006321) z^-1 + 0.382 (+/- 0.009364) z^-2
             + 0.06379 (+/- 0.01485) z^-3


 F(z) = 1 - 1.449 (+/- 0.01355) z^-1 + 1.386 (+/- 0.01449) z^-2
             - 0.3721 (+/- 0.01326) z^-3

Name: oe331
Sample time: 1 seconds

Parameterization:
   Polynomial orders:   nb=3   nf=3   nk=1
   Number of free coefficients: 6

Estimated using PEM on time domain data "mydatade".
Fit to estimation data: 96.03%
FPE: 0.009795, MSE: 0.009562
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Parameter uncertainty is very small. The model fit is shown in red in Fig. 5. Residuals are in Fig. 6 and are within ranges. Zeros/poles are in Fig. 7. The

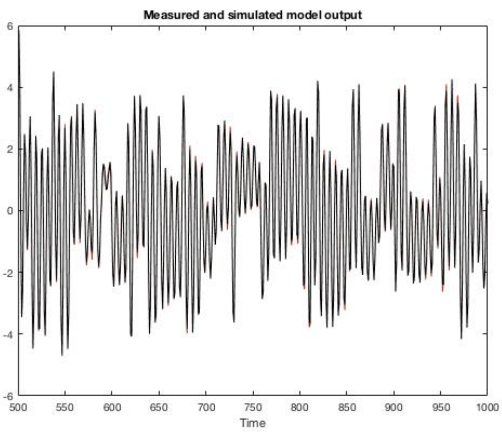

Figure 5: Model fit for OE(3,3,1).

poles are all stable. The confidence intervals are very small. Needless to say, the estimated Bode plot fits the frequency functions estimated with ETFE and SPA, see Fig. 8

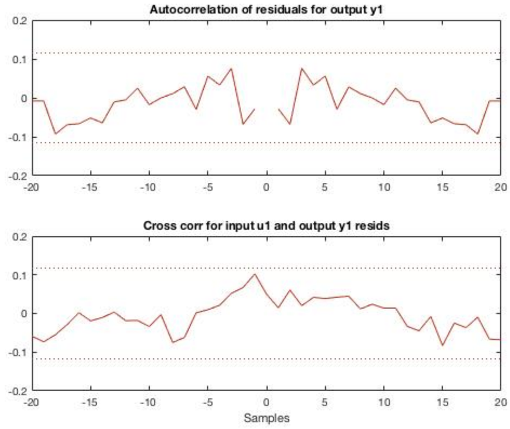

Figure 6: Residuals

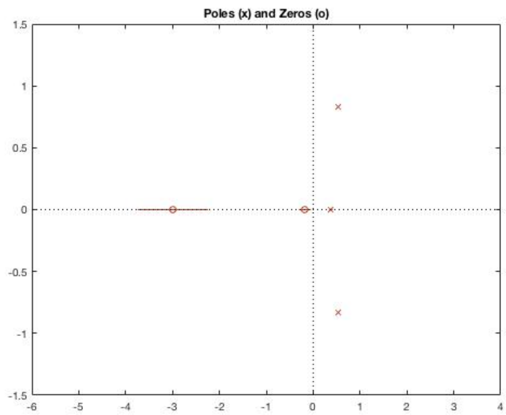

Figure 7: Zeros and poles

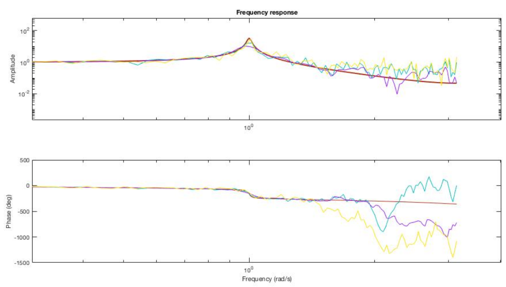

Figure 8: Bode plot of the OE model (thick brown like) and frequency function

4. Exercise 4: bond graph

   (a) The bond graph is given in Fig. 9 and its causality is conflict-free.

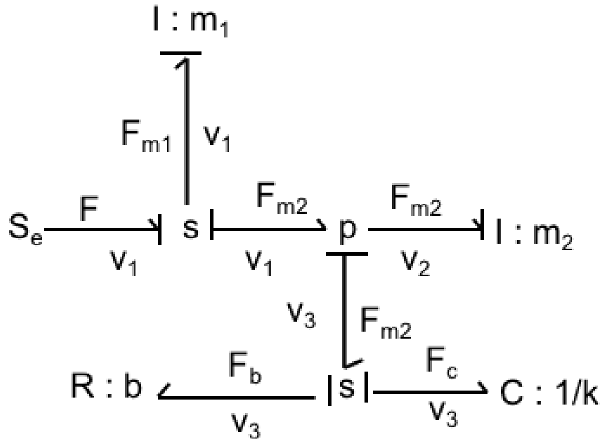

Figure 9: Exercise 4, bond graph

   (b) Let us choose F as input and as state variables the effort variables at the C-elements and the flow variables at the I-elements:

$$x = \begin{pmatrix} v_1 & v_2 & F_c \end{pmatrix}^T$$

   With the choice of variable names shown in Fig. 9 one gets

$$\begin{array}{rcl} \text{s:} & F = & = F_{m1} + F_{m2} \\ \text{I:} & m_1\dot{v}_1 & = F_{m1} \\ \text{p:} & v_3 & = v_1 - v_2 \\ \text{I:} & m_2\dot{v}_2 & = F_{m2} \\ \text{s:} & F_{m2} & = F_c + F_b \\ \text{C:} & 1/k\dot{F}_c & = v_3 \\ \text{R:} & F_b & = bv_3 \end{array}$$

   which yields the following ODEs

$$\begin{aligned} \dot{v}_1 &= \frac{1}{m_1}\left(F - F_c - b(v_1 - v_2)\right) \\ \dot{v}_2 &= \frac{1}{m_2}\left(F_c + b(v_1 - v_2)\right) \\ \dot{F}_c &= k(v_1 - v_2) \end{aligned}$$

   In matrix form then,

$$\dot{x} = \begin{pmatrix} -b/m_1 & b/m_1 & -1/m_1 \\ b/m_2 & -b/m_2 & 1/m_2 \\ k & -k & 0 \end{pmatrix} x + \begin{pmatrix} 1/m_1 \\ 0 \\ 0 \end{pmatrix} F$$

In the spring $F_c = k \int_0^t v_3(s)ds$, hence in terms of y: $F_c = -ky$, meaning that the output equation is

$$y = \begin{pmatrix} 0 & 0 & -1/k \end{pmatrix} x$$

5. Exercise 5: DAE.

The DAE system is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{u} \end{bmatrix} + \begin{bmatrix} 1 & 0 & -1 \\ -2 & 3 & 1 \\ -1 & 0 & D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} y \tag{1}$$

(a) If $D = 1$, differentiating the 3rd eq in (1) and regrouping

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{u} \end{bmatrix} + \begin{bmatrix} 1 & 0 & -1 \\ -2 & 3 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \dot{y}$$

Since the left-most matrix in invertible, the differentiability index is 1.

(b) If $D = 0$ differentiating the 3rd eq in (1) and regrouping

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{u} \end{bmatrix} + \begin{bmatrix} 1 & 0 & -1 \\ -2 & 3 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \dot{y}$$

in which the left-most matrix has rank 2. Summing the first equation to the 3rd and differentiating again the 3rd

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{u} \end{bmatrix} + \begin{bmatrix} 1 & 0 & -1 \\ -2 & 3 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \ddot{y}$$

in which the left-most matrix is now invertible, hence the differentiability index is 2.

(c) Computing the transfer function

$$G(s) = \frac{-1}{s+1} + D = \begin{cases} \frac{s}{s+1} & \text{in case (a)} \\ \frac{-1}{s+1} & \text{in case (b)} \end{cases}$$

meaning that

- case (a): $n = 1$, $m = 1$, diff. index =1
- case (b): $n = 1$, $m = 0$, diff. index =2

Hence in both cases

$$\text{diff. index} = n - m + 1$$