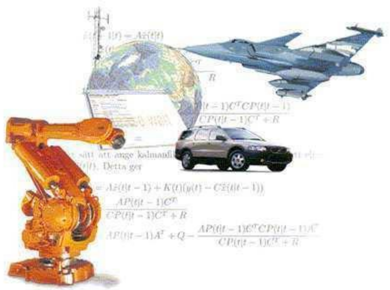
Lecture 2
Inverse kinematics, velocity kinematics and the manipulator Jacobian



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* Substantial contributions to the lecture notes from Johanna Wallén which is gratefully acknowledged.

Previous lecture ...

- p^0 = coordinates for point p relative to coordinate frame 0
- R_1^0 = rotation matrix giving the orientation of frame 1 w.r.t. frame 0

$$R_1^0 = \begin{pmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 & z_1 \cdot x_0 \\ x_1 \cdot y_0 & y_1 \cdot y_0 & z_1 \cdot y_0 \\ x_1 \cdot z_0 & y_1 \cdot z_0 & z_1 \cdot z_0 \end{pmatrix}$$

$$R_0^1 = (R_1^0)^{-1} = (R_1^0)^T$$

Previous lecture ...

- Composition of rotations, R:

  $R_2^0 = R_1^0 R = R_1^0 R_2^1$, performed rotation relative to the current frame, **postmultiply** by R
  $R_2^0 = R R_1^0$, performed rotation relative to the fixed (original) frame, **premultiply** by R

Previous lecture ...

- Composition of rotations, R:

  $R_2^0 = R_1^0 R = R_1^0 R_2^1$, performed rotation relative to the current frame, **postmultiply** by R
  $R_2^0 = R R_1^0$, performed rotation relative to the fixed (original) frame, **premultiply** by R
- Express a rotation in one frame in another frame, **similarity transformation.** A the rotation in frame 0, B the corresponding rotation expressed in frame 1:

$$B = (R_0^1)^T A R_0^1$$

Short summary of previous lecture

- A rigid motion is described by

$$p^0 = R_1^0 p^1 + d^0$$

Short summary of previous lecture

- A rigid motion is described by

$$p^0 = R_1^0 p^1 + d^0$$

Using the homogeneous transformation

$$H_1^0 = \begin{pmatrix} R_1^0 & d^0 \\ 0 & 1 \end{pmatrix}$$

and the homogeneous representations of p^0, p^1

$$\begin{pmatrix} p^0 \\ 1 \end{pmatrix} = P^0, \qquad \begin{pmatrix} p^1 \\ 1 \end{pmatrix} = P^1$$

it gives the homogeneous matrix equation

$$P^0 = H_1^0 P^1$$

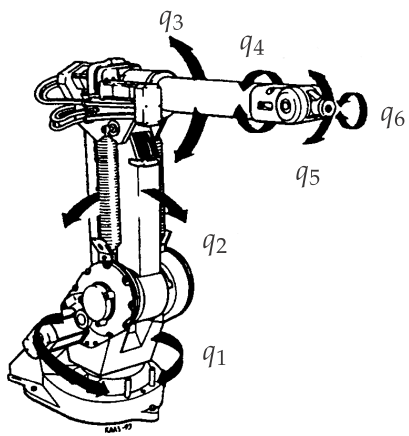
Todays agenda



1. Forward kinematics (Denavit-Hartenberg convention)
2. Inverse kinematics
3. Velocity kinematics
   - Linear velocity
   - Angular velocity
4. What does a Jacobian tell us?
   - Singularities
   - Redundancy



Introduction to robotics



An manipulator has

- *links*
- *joints*. Two “basic” types with one single degree of freedom:
  - revolute
  - prismatic

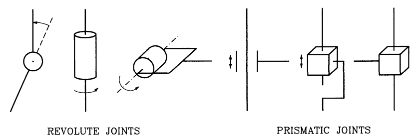
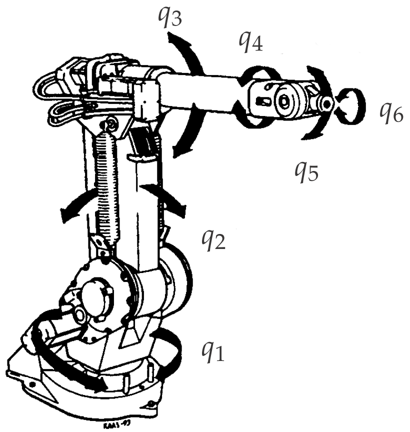


Figure 2.12 in Sciavicco et al.

Introduction to robotics



An manipulator has

- *links*
- *joints*. Two “basic” types with one single degree of freedom:
  - revolute
  - prismatic
- ... and *joint variables* q (here: q_i = joint angles θ_i, $i = 1, \ldots, 6$).

Pose = position and orientation of the end effector (tool)
Configuration = joint variables q in a specified pose

Introduction to robotics

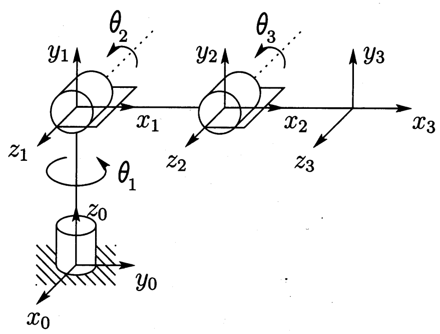


Figure 3.1 in Spong et al.

- A robot with $1, \ldots, n$ joints has $0, \ldots, n$ links ($n+1$ number of links).
- The link 0 is fixed to the ground.
- Coordinate frame i ($\{x_i, y_i, z_i\}$) is associated to link i. ($n+1$ frames)

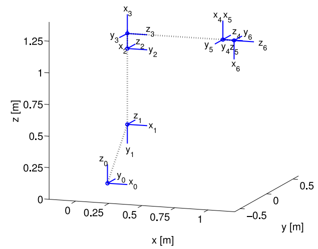
Introduction to robotics

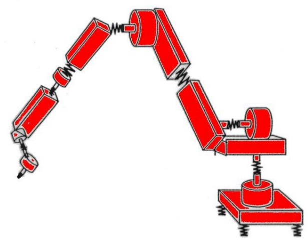
Typically only the joint angles are measured in standard commercial industrial robots.

Actual end effector (or tool) position and orientation are calculated from models of the kinematics.

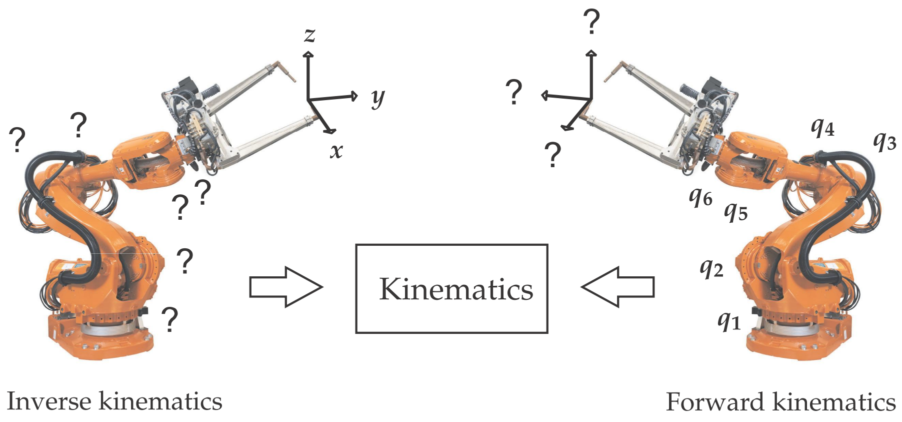
Kinematics and dynamics

- **Kinematics**:
  position, velocity, acceleration.
  Main focus of this lecture.
- **Dynamics**:
  forces and torques.
  Covered in lecture 3!





Forward and inverse kinematics



Position kinematics

Denavit-Hartenberg (D-H) convention – established standard to derive the general kinematics relations. Uses open kinematic chains.

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Denavit-Hartenberg (D-H) convention – established standard to derive the general kinematics relations. Uses open kinematic chains.

Open kinematic chain (serial robot):

Closed kinematic chain (parallel robot): a sequence of links forms a loop

Denavit-Hartenberg convention

A systematic way to determine the homogeneous transformations A_i.

- Homogeneous transformation from frame i to frame $i-1$

$$A_i(q_i) = \begin{pmatrix} R_i^{i-1} & d_i^{i-1} \\ 0 & 1 \end{pmatrix}, \qquad T_n^0 = A_1(q_1)\cdots A_n(q_n)$$

Denavit-Hartenberg convention

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- Every transformation A_i is characterised by

$$\begin{aligned} A_i &= \text{Rot}_{z_{i-1},\theta_i}\text{Trans}_{z_{i-1},d_i}\text{Trans}_{x_i,a_i}\text{Rot}_{x_i,\alpha_i} \\ &= \begin{pmatrix} \cos\theta_i & -\sin\theta_i\cos\alpha_i & \sin\theta_i\sin\alpha_i & a_i\cos\theta_i \\ \sin\theta_i & \cos\theta_i\cos\alpha_i & -\cos\theta_i\sin\alpha_i & a_i\sin\theta_i \\ 0 & \sin\alpha_i & \cos\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

Denavit-Hartenberg convention

The procedure derives the coordinate frames for a rigid body.
Choose: z_i = axis of actuation of joint $i+1$.

The coordinate frames are characterized by:
(D-H1) the axis x_i is perpendicular to the axis z_{i-1}
(D-H2) the axis x_i intersects the axis z_{i-1}

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(D-H1) the axis x_i is perpendicular to the axis z_{i-1}
(D-H2) the axis x_i intersects the axis z_{i-1}

Under these conditions, there exist unique numbers a, d, θ, α (within a multiple 2π) such that

$$A_i = \text{Rot}_{z_{i-1},\theta_i}\text{Trans}_{z_{i-1},d_i}\text{Trans}_{x_i,a_i}\text{Rot}_{x_i,\alpha_i}$$

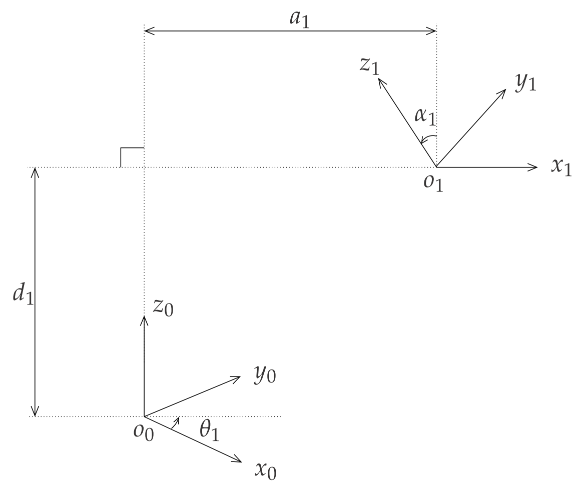
4 parameters are sufficient to specify an arbitrary homogeneous transformation satisfying (D-H1) and (D-H2).

Denavit-Hartenberg parameters

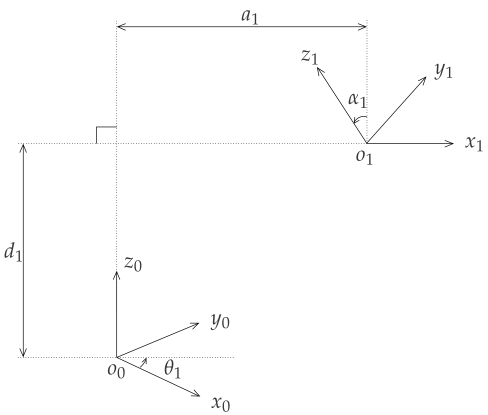
$\theta_i, d_i, a_i, \alpha_i$ are four characteristic D-H parameters, associated with link i and joint i.

For the two cases of joints:

$$q_i = \begin{cases} \theta_i, & \text{revolute} \\ d_i, & \text{prismatic} \end{cases}$$



Denavit-Hartenberg parameters



$$A_i = \text{Rot}_{z_{i-1},\theta_i}\text{Trans}_{z_{i-1},d_i}\text{Trans}_{x_i,a_i}\text{Rot}_{x_i,\alpha_i}$$

- *Angle* θ_i: angle between the x_{i-1} and x_i-axis measured in the plane perpendicular to the z_{i-1}-axis.
- *Offset* d_i: distance between origin o_{i-1} and the intersection of the x_i-axis with z_{i-1}-axis measured along the z_{i-1}-axis.
- *Length* a_i: distance from origin o_i to the intersection between the x_i and z_{i-1}-axis measured along the x_i-axis.
- *Twist* α_i: angle between the z_{i-1} and z_i-axis measured in the plane perpendicular to the x_i-axis.

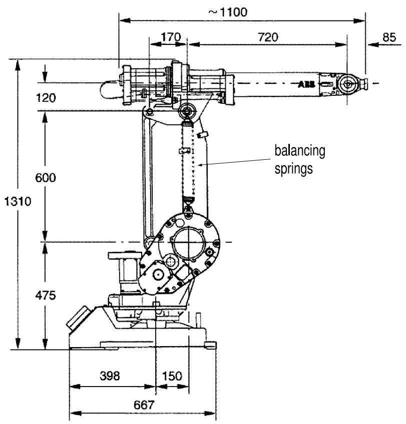
Example: IRB1400

Robot in the research lab, IRB1400:

- Only revolute joints.
- Joint 2 and 3 mechanically coupled.

Example: IRB1400

Robot in the research lab, IRB1400:



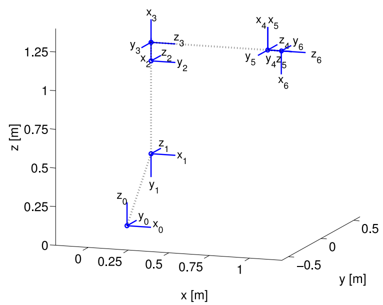
- Only revolute joints.
- Joint 2 and 3 mechanically coupled.
- Possible to rewrite to a serial structure. D-H joint variables θ given by

$$\theta = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \phi + \begin{pmatrix} 0 \\ -pi/2 \\ 0 \\ 0 \\ 0 \\ \pi \end{pmatrix}$$

- D-H parameters

$$d = \begin{pmatrix} 0.475 & 0 & 0 & 0.72 & 0 & 0.085 \end{pmatrix}$$
$$a = \begin{pmatrix} 0.15 & 0.6 & 0.12 & 0 & 0 & 0 \end{pmatrix}$$
$$\alpha = \begin{pmatrix} -\pi/2 & 0 & -\pi/2 & \pi/2 & -\pi/2 & 0 \end{pmatrix}$$

Example: IRB1400

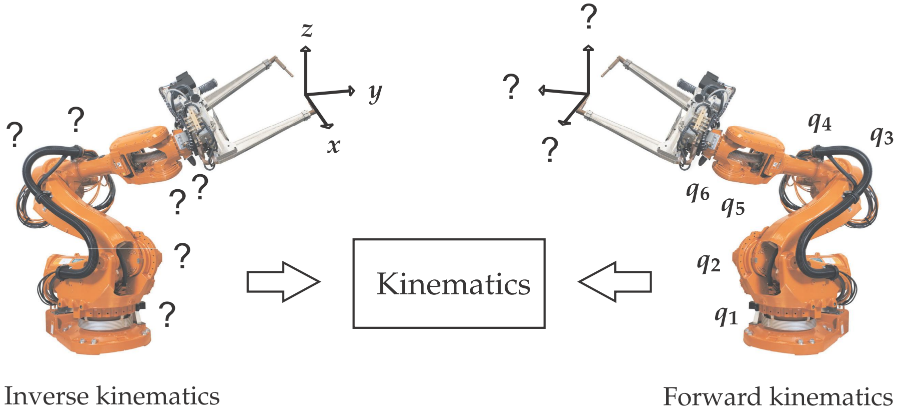


Robotics Toolbox (Matlab):

- ```
  L(1) = Link([theta(1), d(1), a(1),
  alpha(1)]);
  ...
  L(6) = Link([theta(6), d(6), a(6),
  alpha(6)]);
  ```
- ```
  robot6 = SerialLink(L);
  q = [0 -pi/2 0 0 0 pi];
  plot(robot6, q)
  ```

Inverse kinematics

Derive the joint variables $q_1, \ldots, q_n$, when the end effector position and orientation are known. Generally a harder problem than the forward kinematics.



Inverse kinematics

Given the homogeneous transformation $H = \begin{pmatrix} R & o \\ 0 & 1 \end{pmatrix} \in SE(3)$, find a solution (possibly several solutions) to

$$T_n^0(q_1, \ldots, q_n) = H, \quad \text{where} \quad T_n^0(q_1, \ldots, q_n) = A_1(q_1) \ldots A_n(q_n)$$

This gives the equations

$$T_{ij}(q_1, \ldots, q_n) = h_{ij}, \quad i = 1,2,3, \quad j = 1,2,3,4$$

Hard to solve in closed form. Must use the actual kinematic structure to simplify the problem.

Inverse kinematics – solutions?

The existence of solutions to the inverse kinematics problem depends on engineering as well as mathematical considerations.

Example:

The motion of a joint can be restricted to less than $360°$.

Not all mathematical solutions to the kinematic equations correspond to robot configurations that are physically realisable.

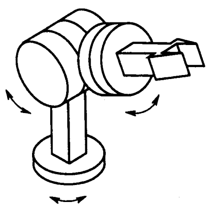


Figure 1.7 in Sciavicco et al.

Inverse kinematics – kinematic decoupling

Assume that we have a 6 DOF robot with a spherical wrist. It means that the robot has 6 joints, where the 3 joint axes of the wrist intersect at a point (called wrist center o_c).

$\Rightarrow$ possible to decouple into two simpler problems:

- Inverse *position* kinematics
- Inverse *orientation* kinematics

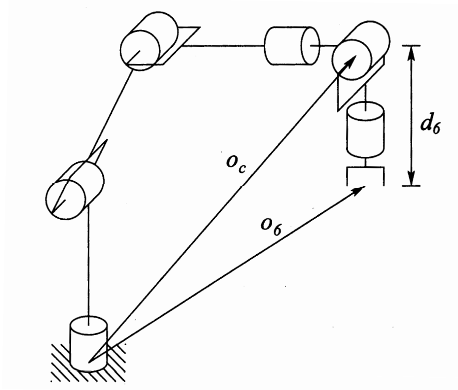


Figure 3.12 in Spong et al.

Inverse position – geometric approach

Inverse position: find the joint variables q_1, q_2, q_3 corresponding to a given position of the wrist center o_c.

General idea:
Solve for joint variable q_i by projecting onto the x_{i-1}, y_{i-1}-plane.
Results in a simple trigonometric problem.

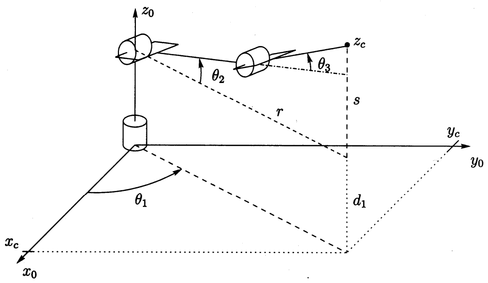


Figure 3.13 in Spong et al.

Inverse position – multiple solutions

Multiple solutions can be found.

Left and right arm configuration

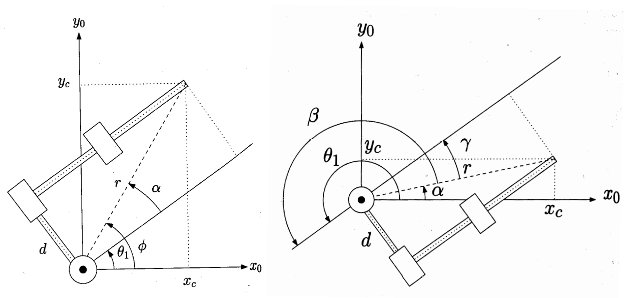


Figure 3.17 in Spong et al.

PUMA manipulator

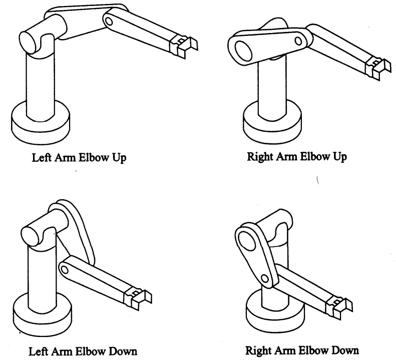


Figure 3.18 in Spong et al.

Inverse orientation – geometric approach

Inverse orientation: find the joint variables q_4, q_5, q_6 corresponding to a given orientation with respect to the frame $\{x_3 y_3 z_3\}$.

Example:
For a spherical wrist it means to find the set of Euler angles ϕ, θ, ψ corresponding to a given rotation matrix R. Then use the mapping

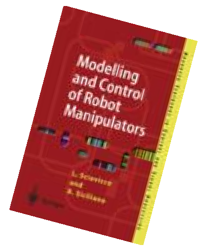
$$\theta_4 = \phi, \quad \theta_5 = \theta, \quad \theta_6 = \psi$$

No general approach when solving the inverse kinematics problem is given. Special treatment of every single type of kinematic structure of the robot.

Summary – position kinematics

A good summary of the forward kinematics problem using the Denavit-Hartenberg convention and the inverse kinematics problem.
Spong et al., pages 110-111.

The Denavit-Hartenberg convention for a closed kinematic chain is described.
Sciavicco & Siciliano, pages 46-49.



Velocity kinematics – the Jacobian

The (manipulator) Jacobian J relates the linear velocity v_n^0 and angular velocity ω_n^0 of the end effector to the derivative of the joint variables q

$$\begin{pmatrix} v_n^0 \\ \omega_n^0 \end{pmatrix} = J(q)\dot{q}.$$

The Jacobian is one of the most important quantities in robot analysis and control!

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The Jacobian is one of the most important quantities in robot analysis and control!

- Planning of trajectories
- Determination of singular configurations
- Analysis of redundancy
- Derivation of dynamic equations of motion
- Transformation of forces and torques from the end effector to the robot joints
- · · ·

Repition: Angular velocity

- Rotation of an angle θ about a fix axis k.
  Angular velocity
  $$\omega = \dot{\theta}k$$
- Linear velocity of any point on the body.
  $$v = \omega \times r, \quad r = \text{ the vector from the origin to the point}$$

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  $$\omega = \dot{\theta}k$$
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  $$v = \omega \times r, \quad r = \text{ the vector from the origin to the point}$$
- Resulting angular velocity due to relative rotation of several coordinate frames. Angular velocities added as free vectors if they are expressed relative to the same frame.
  $$\begin{aligned} \omega_{0,n}^0 &= \omega_{0,1}^0 + R_1^0\omega_{1,2}^1 + \ldots + R_{n-1}^0\omega_{n-1,n}^{n-1} \\ &= \omega_{0,1}^0 + \omega_{1,2}^0 + \ldots + \omega_{n-1,n}^0 \end{aligned}$$

Repition: Skew symmetric matrix

- S is skew symmetric if $S^T + S = 0$. With $s = \begin{pmatrix} s_1 & s_2 & s_3 \end{pmatrix}^T$ we define

$$S(s) = \begin{pmatrix} 0 & -s_3 & s_2 \\ s_3 & 0 & -s_1 \\ -s_2 & s_1 & 0 \end{pmatrix}$$

- Derivative of rotation matrix

$$\frac{d}{dt}R(t) = S(\omega(t))R(t), \quad \text{solution } R(t) = e^{S(\omega)t}R(0)$$

$\omega(t)$ = angular velocity of the rotating frame w.r.t. the fixed frame at time t

Expressions for relative velocity transformations between coordinate frames involve derivatives of rotation matrices.

Derivation of the Jacobian

We have an n-link robot with joint variables $q_1, \ldots, q_n$. The transformation from end effector frame n to base frame 0

$$T_n^0(q) = \begin{pmatrix} R_n^0(q) & o_n^0 \\ 0 & 1 \end{pmatrix}$$

o_n^0 = end effector position expressed in frame 0
$R_n^0(q)$ = end effector orientation

The angular velocity of the end effector: $S(\omega_n^0) = \dot{R}_n^0(R_n^0)^T$.
The linear velocity of the end effector: $v_n^0 = \dot{o}_n^0$.

The Jacobian is given by

$$\begin{pmatrix} v_n^0 \\ \omega_n^0 \end{pmatrix} = \begin{pmatrix} J_v \\ J_\omega \end{pmatrix} \dot{q}, \quad \text{also denoted } \xi = J\dot{q}$$

Derivation of the Jacobian – linear velocity

The linear velocity of the end effector is $v_n^0 = \dot{o}_n^0$. The chain rule gives

$$\dot{o}_n^0 = \sum_{i=1}^{n} \frac{\partial o_n^0}{\partial q_i}\dot{q}_i$$

The ith column of J_v is $J_{v_i} = \frac{\partial o_n^0}{\partial q_i}$. The linear velocity of the end effector if $\dot{q}_i = 1$ and $\dot{q}_j = 0, \quad j \neq i$.

$$v_n^0 = \begin{cases} \omega_n^0 \times r = \dot{q}_i z_{i-1}^0 \times (o_n^0 - o_{i-1}^0), & \text{revolute joint } (q_i = \theta_i) \\ \dot{q}_i R_{i-1}^0 z_{i-1}^{i-1} = \dot{q}_i z_{i-1}^0, & \text{prismatic joint } (q_i = d_i) \end{cases}$$

This results in

$$J_v = \begin{pmatrix} J_{v_1} & \ldots J_{v_n} \end{pmatrix}, \quad \text{where } J_{v_i} = \begin{cases} z_{i-1}^0 \times (o_n^0 - o_{i-1}^0), & \text{revolute} \\ z_{i-1}^0, & \text{prismatic} \end{cases}$$

Derivation of the Jacobian – angular velocity

The angular velocity of link i resulting from rotation of joint i, expressed in frame $i-1$

$$\omega_i^{i-1} = \begin{cases} \dot{q}_i z_{i-1}^{i-1} = \dot{q}_i k, & \text{revolute} \\ \omega_i^{i-1} = 0, & \text{prismatic} \end{cases}$$

The overall angular velocity of the end effector in the base frame 0 by adding the results from each single link expressed in frame 0

$$\omega_n^0 = \rho_1\dot{q}_1 k + \rho_2\dot{q}_2 R_1^0 k + \ldots + \rho_n\dot{q}_n R_{n-1}^0 k$$
$$= \sum_{i=1}^{n} \rho_i\dot{q}_i z_{i-1}^0, \quad \rho = \begin{cases} 1, & \text{revolute} \\ 0, & \text{prismatic} \end{cases}$$

This gives $J_\omega = \begin{pmatrix} \rho_1 z_0^0 & \ldots & \rho_n z_{n-1}^0 \end{pmatrix}$

Example:

Robotics Toolbox:

- `robot6 = SerialLink(L);`
- `qz = [0 -pi/2 0 0 0 pi];`
- `J := robot6.jacob0(qz);`

The analytical Jacobian

The *analytical Jacobian* is based on the minimal representation for the orientation of the end effector frame.

The end effector pose

$$X = \begin{pmatrix} d(q) \\ \alpha(q) \end{pmatrix}$$

$d(q)$ = usual vector from the origin of the base frame 0 to the origin of the end effector frame n.
$\alpha(q)$ = minimal representation for the orientation of the end effector frame n relative to the base frame 0. For example $\alpha = \begin{pmatrix} \phi & \theta & \psi \end{pmatrix}^T$.

The analytical Jacobian is defined by

$$\dot{X} = \begin{pmatrix} \dot{d}(q) \\ \dot{\alpha}(q) \end{pmatrix} = J_a(q)\dot{q}$$

The analytical Jacobian

Geometric Jacobian $\xi = \begin{pmatrix} v(q) \\ \omega(q) \end{pmatrix} = \begin{pmatrix} \dot{d}(q) \\ \omega(q) \end{pmatrix} = J(q)\dot{q}$

Analytical Jacobian $\dot{X} = \begin{pmatrix} \dot{d}(q) \\ \dot{\alpha}(q) \end{pmatrix} = J_a(q)\dot{q}$

Assume that the system is transformed by the Euler angle transformation, where α = Euler angles. It gives $\omega = B(\alpha)\dot{\alpha}$.

The relation between the geometric and analytical Jacobian is then

$$J(q)\dot{q} = \begin{pmatrix} v(q) \\ \omega(q) \end{pmatrix} = \begin{pmatrix} \dot{d}(q) \\ B(\alpha)\dot{\alpha} \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & B(\alpha) \end{pmatrix} J_a(q)\dot{q}$$

(See Sciavicco et al., Chapter 3.6 and Spong et al., Chapter 4.8.)

Inverse velocity and acceleration

The inverse velocity/acceleration: find the joint velocities $\dot{q}$ that produce the desired end effector velocity $\dot{X}$ or acceleration $\ddot{X}$.

Differentiating $\dot{X} = J_a(q)\dot{q}$ gives an expression for the acceleration

$$\ddot{X} = J_a(q)\ddot{q} + \left(\frac{d}{dt}J_a(q)\right)\dot{q}$$

For a 6 DOF-robot the inverse velocity and acceleration are

$$\dot{q} = J_a(q)^{-1}\dot{X}$$
$$\ddot{q} = J_a(q)^{-1}\left(\ddot{X} - \left(\frac{d}{dt}J_a(q)\right)\dot{q}\right)$$

provided $\det J_a(q) \neq 0$.

Singularities

The $6 \times n$ Jacobian $J(q)$ defines a (time-varying) mapping $\xi = J(q)\dot{q}$. All possible end effector velocities are linear combinations of the columns of J

$$\xi = J_1\dot{q}_1 + J_2\dot{q}_2 + \ldots + J_n\dot{q}_n$$

When rank $J = 6$, the end effector can execute an arbitrary velocity $\xi \in \mathbb{R}^6$. For $J \in \mathbb{R}^{6\times 6}$, the Jacobian looses rank when $\det J = 0$.

The rank is configuration dependent. Configurations for which the rank is less than the maximal value are called (kinematic) *singularities* or *singular configurations*.

Singularities

Example: Two-link planar arm.

The Jacobian is given by

$$J(q) = \begin{pmatrix} -a_1\sin\theta_1 - a_2\sin(\theta_1+\theta_2) & -a_2\sin(\theta_1+\theta_2) \\ a_1\cos\theta_1 + a_2\cos(\theta_1+\theta_2) & a_2\cos(\theta_1+\theta_2) \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix}$$

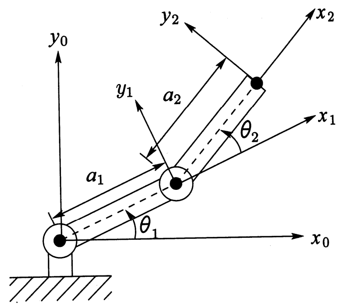

Figure 3.6 in Spong et al.

Singularities

Singularities can be classified into:

- *Boundary singularities*. When the robot is completely outstretched or retracted. Can be avoided by the condition that the robot is not driven to the boundaries of its workspace.
- *Internal singularities*. Occur inside the reachable workspace. Generally caused by the alignment of two or more axes of motion. A serious problem, since they can be encountered anywhere in the reachable workspace for a planned path.

Singularities

Examples of internal singularities, where two or more axes are aligned.

Elbow singularity

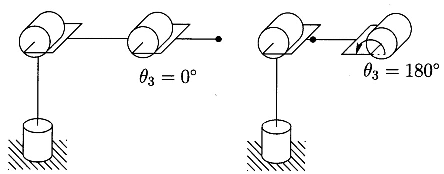

Figure 4.6 in Spong et al.

Elbow fully extended or retracted.

Spherical wrist singularity

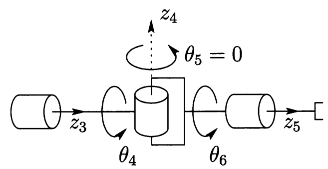

Figure 4.4 in Spong et al.

Axes z_3 and z_5 collinear.

Singularities

Identifying the singularities are *very important* in robotic applications!

- Singularities represent configurations from which certain directions of motion may be unattainable.
- Bounded end effector velocities may correspond to unbounded joint velocities at singularities.
- Bounded joint torques may correspond to unbounded end effector forces and torques at singularities.
- Singularities correspond to points in the robot workspace that may be unreachable under small perturbations of the link parameters.

Singularities – decoupling

Assume that we have the Jacobian $J \in \mathbb{R}^{6\times 6}$. Singular configurations q are given by $\det J = 0$. Generally a hard problem to solve.

For robots with a spherical wrist it is possible to decouple the singularities into

- *Arm singularities*. Resulting from the motion of the arm.
- *Wrist singularities*. Resulting from motion of the spherical wrist.

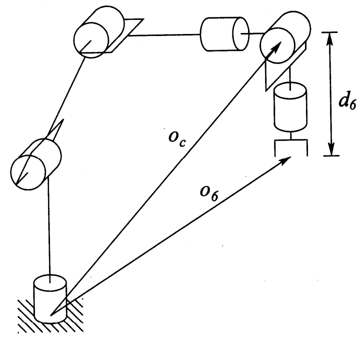

Figure 3.12 in Spong et al.

Example: Singularities – decoupling

Example: 6 DOF robot, with 3 DOF-arm and 3 DOF-spherical wrist.

$J \in \mathbb{R}^{6\times 6}$. Partition the Jacobian into

$$J = \begin{pmatrix} J_P & J_O \end{pmatrix} = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix}$$

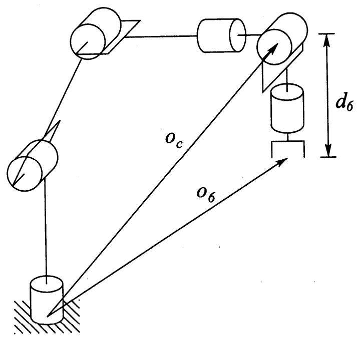

Figure 3.12 in Spong et al.

Example: Singularities – decoupling

Example: 6 DOF robot, with 3 DOF-arm and 3 DOF-spherical wrist.

$J \in \mathbb{R}^{6\times 6}$. Partition the Jacobian into

$$J = \begin{pmatrix} J_P & J_O \end{pmatrix} = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix}$$

Since the final three joints are revolute

$$J_O = \begin{pmatrix} z_3 \times (o_6 - o_3) & z_4 \times (o_6 - o_4) & z_5 \times (o_6 - o_5) \\ z_3 & z_4 & z_5 \end{pmatrix}$$

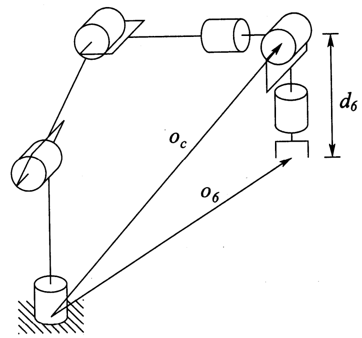

Figure 3.12 in Spong et al.

Singularities – decoupling

The wrist axes intersect at a common point o_c. Choosing coordinate frames so that $o_3 = o_4 = o_5 = o_6 = o_c$ gives

$$J_O = \begin{pmatrix} 0 & 0 & 0 \\ z_3 & z_4 & z_5 \end{pmatrix}$$

The Jacobian

$$J = \begin{pmatrix} J_{11} & 0 \\ J_{21} & J_{22} \end{pmatrix}, \quad \det J = \det J_{11} \det J_{22}$$

Singular configurations:
union of *arm configurations* satisfying $\det J_{11} = 0$ and *wrist configurations* satisfying $\det J_{22} = 0$.

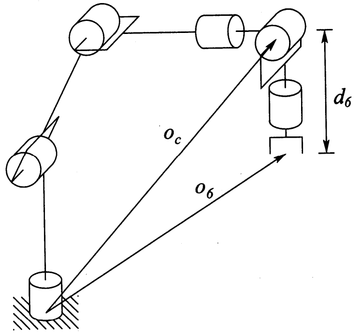

Figure 3.12 in Spong et al.

Singularities – decoupling

Arm singularities ($\det J_{11} = 0$) when $\theta_3 = 0, \pi$.
Also when $a_2 \cos\theta_2 + a_3 \cos(\theta_2 + \theta_3) = 0$, the wrist center o_c intersects the axis z_0.

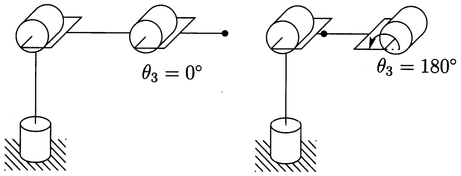

Figure 4.6 in Spong et al.

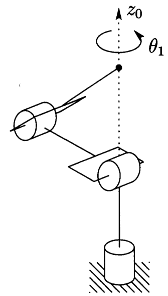

Figure 4.7 in Spong et al.

Singularities – decoupling

Arm singularities ($\det J_{11} = 0$) when $\theta_3 = 0, \pi$.
Also when $a_2 \cos\theta_2 + a_3 \cos(\theta_2 + \theta_3) = 0$, the wrist center o_c intersects the axis z_0.

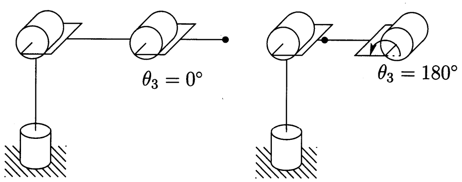

Figure 4.6 in Spong et al.

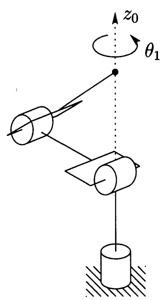

Figure 4.7 in Spong et al.

Wrist singularities when z_3 and z_5 are collinear, from $\det \begin{pmatrix} z_3 & z_4 & z_5 \end{pmatrix} = 0$.

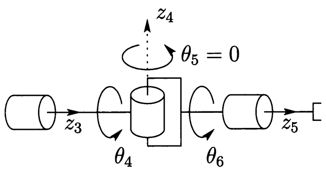

Figure 4.4 in Spong et al.

(Example 4.9 in Spong et al.)

Redundancy

The human arm has 7 degrees of mobility:
Three in the shoulder, one in the elbow and three in the wrist. (Not considered the degrees of mobility in the fingers...)
The arm is *redundant*, since we only can perform motions with 6 degrees of mobility.

Redundancy is an important concept in robotic applications.
Using a redundant robot increases the dexterity and mobility.

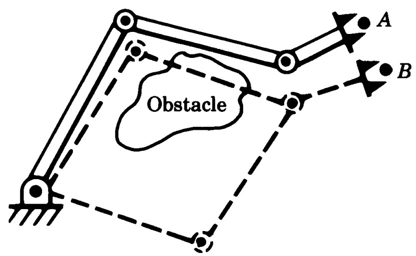

Figure 4.3 in Craig

(See Sciavicco et al., Chapters 2.10.2 and 3.4 for redundancy.)

Redundancy – categorisation

Redundancy of a robot is categorised into:

- *Kinematically redundant*. The degree of mobility of the robot is larger than the number of variables needed to describe the task.
- *Intrinsically redundant*. The dimension of the operational space is smaller than the dimension of the joint space, $m < n$.
- *Functionally redundant*. As an example, when $m = n$, the robot is functionally redundant when only $r < m$ number of components of operational space are of concern for the task.

(Operational space = minimal vector to describe end effector pose, defined in the space in which the robot task is specified.)

Redundancy is thereby a *concept relative to the actual robot task*.

Kinematically redundant

Some notation:
m = number of operational space variables
n = number of degrees of mobility of the kinematic structure
r = number of operational space variables needed to describe the specific task

Study the velocity kinematics

$$\xi = \begin{pmatrix} v \\ \omega \end{pmatrix} = J(q)\dot{q}$$

$\xi = r \times 1$-vector of end effector velocity of concern for the task
$J = r \times n$ Jacobian
$\dot{q} = n \times 1$-vector of joint velocities

If $r < n$, the robot is kinematically redundant. We have $n - r$ redundant degrees of mobility.

Redundancy

The Jacobian defines a linear mapping from the joint velocity space $\dot{q}$ to the end effector velocity space (called v in the figure), for a given pose.

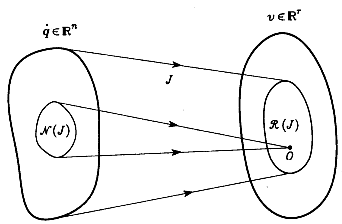

Figure 3.7 in Sciavicco et al.

- The *range* of J: subspace $\mathcal{R}(J) \in \mathbb{R}^r$ of end effector velocities that can be generated by the joint velocities.
- The *null* of J: subspace $\mathcal{N}(J) \in \mathbb{R}^n$ of joint velocities that do not produce any end effector velocity.

Redundancy – reconfiguration

Denote $\dot{q}^*$ a solution to $\xi = J(q)\dot{q}$ and P a $n \times n$-matrix such that $\mathcal{R}(P) \equiv \mathcal{N}(J)$. Then also the joint velocity vector

$$\dot{q} = \dot{q}^* + P\dot{q}_0, \quad \text{arbitrary } \dot{q}_0$$

is a solution to $\xi = J(q)\dot{q}$. Multiplying by J gives

$$J\dot{q} = J\dot{q}^* + JP\dot{q}_0 = J\dot{q}^* = \xi$$

Fundamental importance!
Possible to choose $\dot{q}$ to make use of the redundant degrees of mobility. It generates internal motions that do not change the end effector pose. Can reconfigure the robot into a more dexterous pose for the specific task.

Example: Redundancy

Robotics Toolbox:

- Create the robot

```
clear L
L(1) = Link([ 0 0 0.8 0 ]);
L(2) = Link([ 0 0 1.2 0 ]);
L(3) = Link([ 0 0 1 0 ]);
r3 = SerialLink(L);

L(4) = Link([ 0 0 0.5 0 ]);
r4 = SerialLink(L);
```

- End effector position and Jacobian for a specified pose

```
q = [ 0 1.2 1 0.3 ];
p = transl(fkine(r4, q));
Jq = jacob0(r4,q);
ns = null(Jq)
```


Summary ...

- The structure of a robot (links, joints, chains)
- Kinematics – geometric description and Denavit-Hartenberg convention
- Forward position and velocity kinematics
- Inverse kinematics – decoupling
- The geometric and analytical Jacobian
- Analysis of robot motion – singularities and redundancy.

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Lecture 2: Inverse kinematics, velocity kinematics and the manipulator Jacobian
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