Lecture 1. Rigid body motion.

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Content

- Rigid body transformation
- Rotation
  - Rotation matrices
  - Euler's theorem
  - Parameterization of SO(3)
- Homogeneous representation
  - Matrix representation
  - Chasles' theorem

Background to modeling

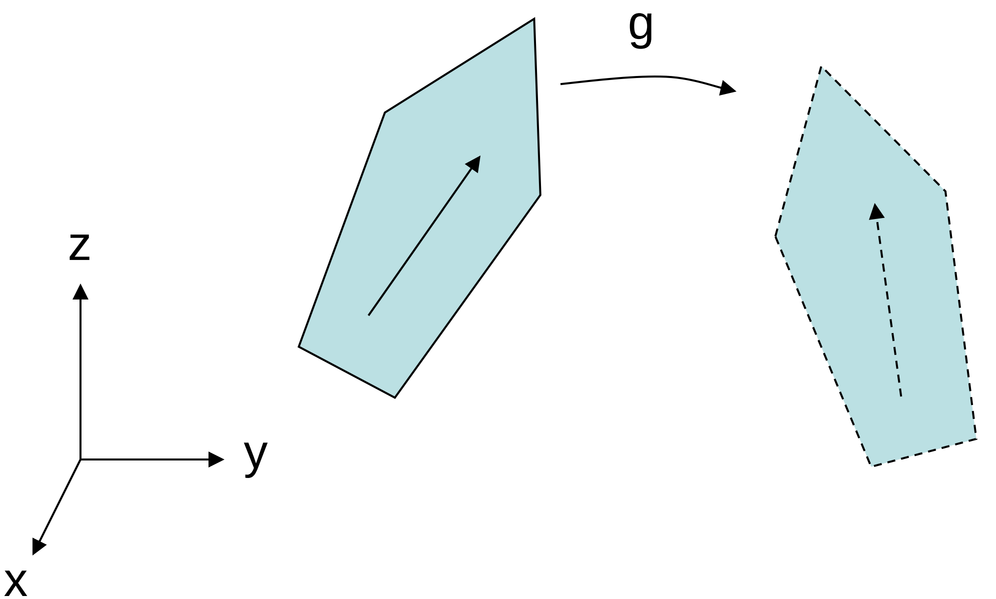
Kinematics

- studies the motion of objects without consideration of the circumstances leading to the motion

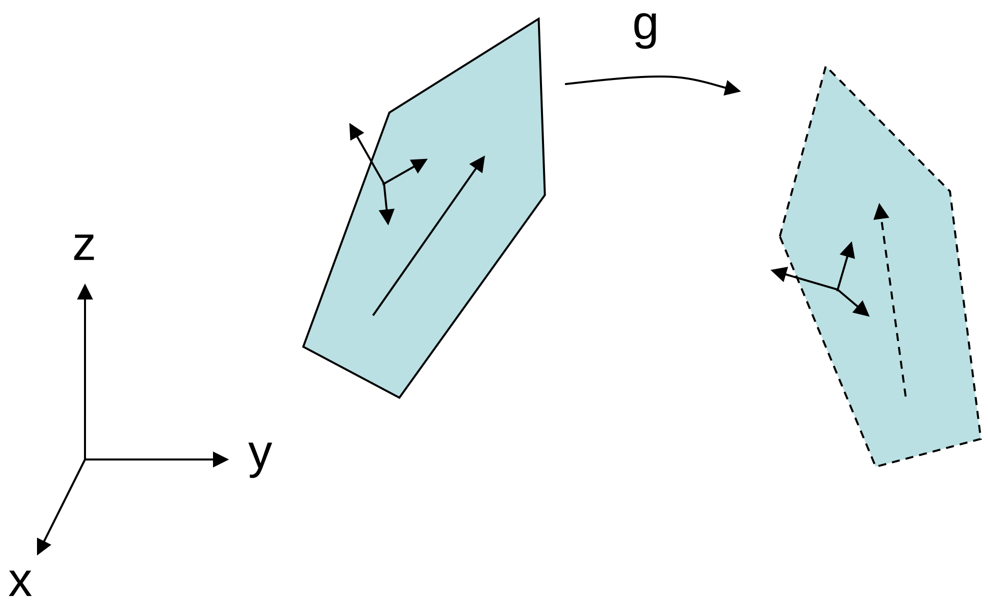
Dynamics

- studies the relationship between the motion of objects and its causes

Rigid body motion



Rigid body motion



The motion of a rigid body can be parameterized as

- position
- orientation

of one point of the object. The ***configuration***.

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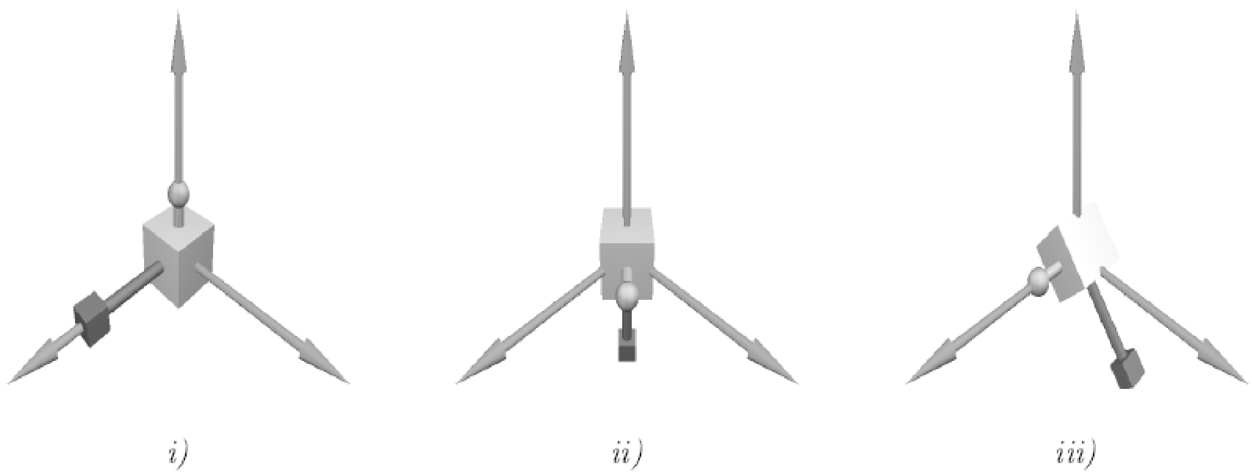
- Homogeneous representation
  - Matrix representation

Representation of orientation

- Rotation matrices
- Angle – axis representation
- Euler angles
- Quaternion
- Exponential coordinates
- …

Composition of rotation

☹ The order of rotation axes is important



$$R_1(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & \sin(\alpha) \\ 0 & -\sin(\alpha) & \cos(\alpha) \end{bmatrix}$$

$$R_2(\alpha) = \begin{bmatrix} \cos(\alpha) & 0 & -\sin(\alpha) \\ 0 & 1 & 0 \\ \sin(\alpha) & 0 & \cos(\alpha) \end{bmatrix}$$

$$R_3(\alpha) = \begin{bmatrix} \cos(\alpha) & \sin(\alpha) & 0 \\ -\sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rx – pi/2

Ry – pi/4

ii) xyz, iii) zyx

Example

Example 2.8

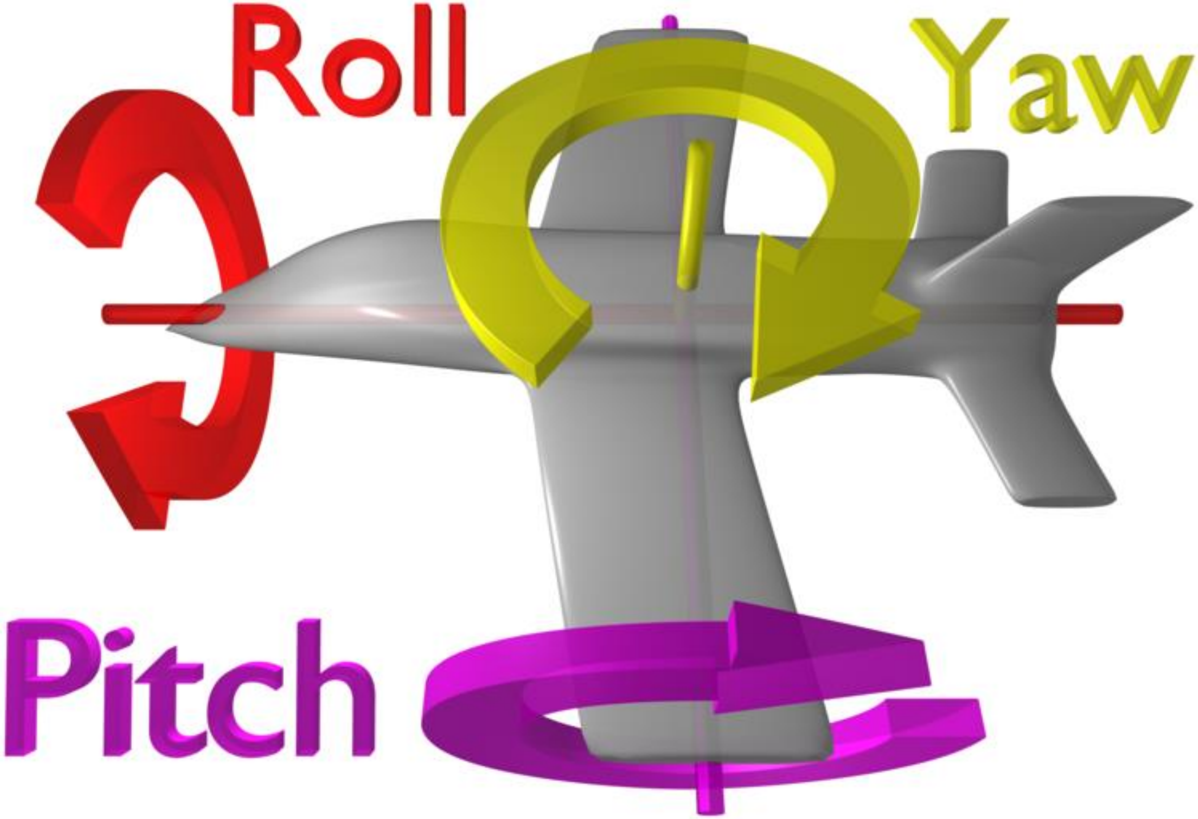
Suppose R *is defined by the following sequence of basic rotations in the order specified:*

1. *A rotation of* θ *about the current* x*-axis*
2. *A rotation of* ϕ *about the current* z*-axis*
3. *A rotation of* α *about the fixed* z*-axis*
4. *A rotation of* β *about the current* y*-axis*
5. *A rotation of* δ *about the fixed* x*-axis*

In order to determine the cumulative effect of these rotations we simply begin with the first rotation $R_{x,\theta}$ *and pre- or post-multiply as the case may be to obtain*

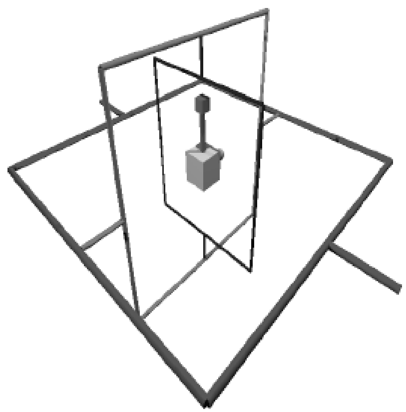
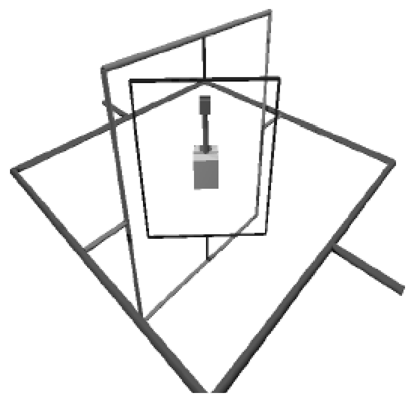
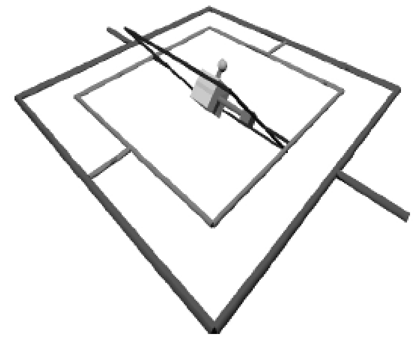
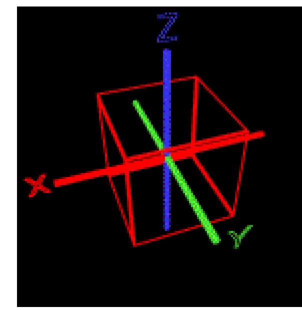
$$R = R_{x,\delta} R_{z,\alpha} R_{x,\theta} R_{z,\phi} R_{y,\beta} \tag{2.24}$$

Euler angles



Euler angles

☹ Gimbal lock (Apollo IMU Gimbal lock 1, 2)

$$
\begin{aligned}
R(\alpha, \tfrac{\pi}{2}, \gamma) &= \begin{pmatrix} 0 & \cos\gamma\sin\alpha - \cos\alpha\sin\gamma & \cos\alpha\cos\gamma + \sin\alpha\sin\gamma & 0 \\ 0 & \cos\alpha\cos\gamma + \sin\alpha\sin\gamma & \cos\alpha\sin\gamma - \cos\gamma\sin\alpha & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & \sin(\alpha-\gamma) & \cos(\alpha-\gamma) & 0 \\ 0 & \cos(\alpha-\gamma) & \sin(\alpha-\gamma) & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\end{aligned}
$$

Euler angles

- ☹ Implementing interpolation is difficult
- ☹ Ambiguous correspondence to rotations
- ☹ The result of composition is not apparent
- ☹ Non-linear dynamics

- ☺ Mathematics is well known
- ☺ Can be visualized “in the mind”

Quaternions

Sir William Rowan Hamilton (1809-1865)

Lectures on Quaternions: Containing a systematic statement of

A New Mathematical Method

of which the principles were communicated in 1843 to the Royal Irish Academy; and which has since formed the subject of successive courses of lectures, delivered in 1848 and subsequent years in the halls of Trinity College, Dublin: with numerous illustrative diagrams, and with some geometrical and physical applications.

Quaternions

Generalization of complex numbers to 3D.

$s + i\,x + j\,y + k\,z$

with $i^2 = j^2 = k^2 = ijk = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$.

A quaternion is usually represented as $q = <s,v>$ with

- s scalar (real part)
- v vector in R^3 (complex part)

Unit quaternion $||q|| = 1$.

Rotation with quaternions

Angle axis to quaternion

$$\theta, v \Rightarrow \quad q = \left\langle \cos\frac{\theta}{2}, \sin\frac{\theta}{2} v \right\rangle$$

Composition of rotations, q_1 then q_2

$$q = q_2 q_1$$

Rotation with quaternions

Rotation of a vector, $u = Rv$

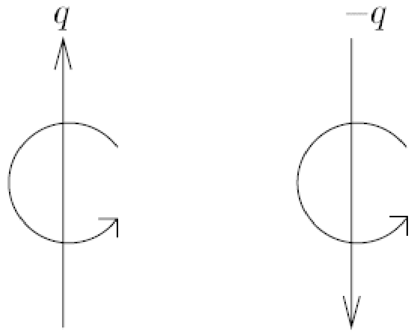
$v_q = \langle 0, v \rangle$, q is quaternion representation of R

$u_q = q v_q q^{-1} = \langle 0, u \rangle$

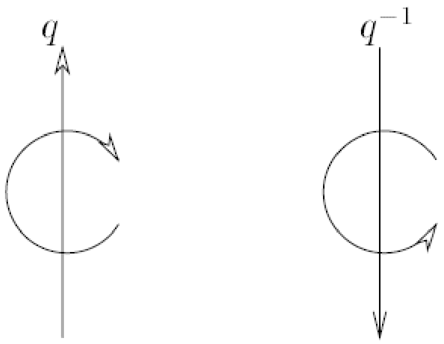
$$R_q(\mathbf{q}) =$$
$$\begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2q_1q_2 + 2q_0q_3 & 2q_1q_3 - 2q_0q_2 \\ 2q_1q_2 - 2q_0q_3 & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2q_2q_3 + 2q_0q_1 \\ 2q_1q_3 + 2q_0q_2 & 2q_2q_3 - 2q_0q_1 & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix}.$$

Some remarks

- *q* and –*q* represent the same rotation



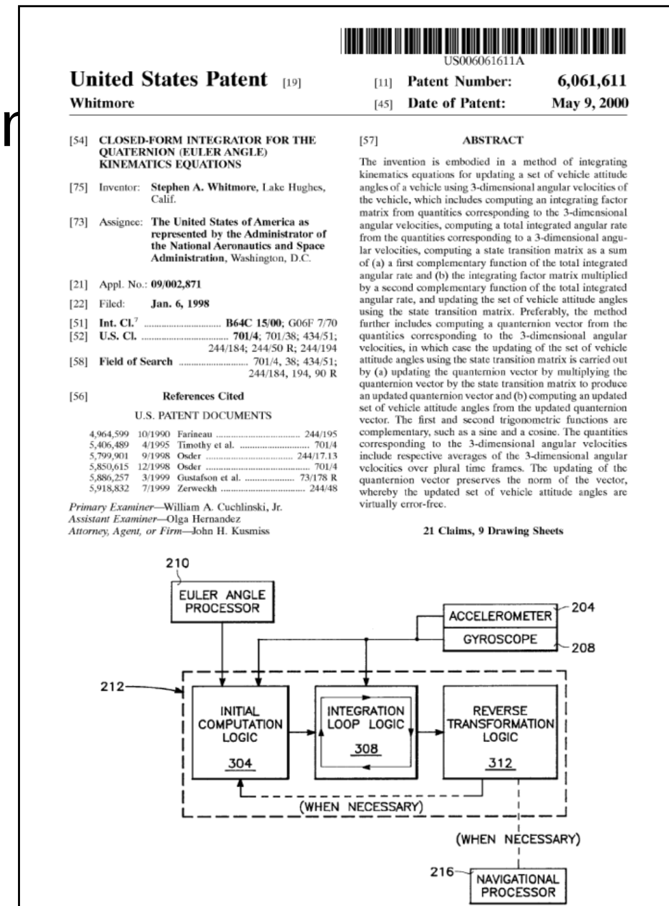
- $q = <s,v>$ and $q^{-1} = <s,-v>$



Quaternions

- ☹ Can only represent orientation
- ☹ Quaternion math is not so well known

- ☺ Compact representation, based upon
- ☺ Simple interpolation methods
- ☺ No gimbal lock
- ☺ Simple composition
- ☺ Linear (bi-linear) dynamics, (NASA)



US006061611A

United States Patent [19]
Whitmore

[11] **Patent Number:** **6,061,611**
[45] **Date of Patent:** **May 9, 2000**

[54] **CLOSED-FORM INTEGRATOR FOR THE QUATERNION (EULER ANGLE) KINEMATICS EQUATIONS**

[75] Inventor: **Stephen A. Whitmore**, Lake Hughes, Calif.

[73] Assignee: **The United States of America as represented by the Administrator of the National Aeronautics and Space Administration**, Washington, D.C.

[21] Appl. No.: **09/002,871**

[22] Filed: **Jan. 6, 1998**

[51] **Int. Cl.**[7] **B64C 15/00**; G06F 7/70
[52] **U.S. Cl.** .... **701/4**; 701/38; 434/51; 244/184; 244/50 R; 244/194
[58] **Field of Search** .... 701/4, 38; 434/51; 244/184, 194, 90 R

[56] **References Cited**

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Primary Examiner—William A. Cuchlinski, Jr.
Assistant Examiner—Olga Hernandez
Attorney, Agent, or Firm—John H. Kusmiss

[57] **ABSTRACT**

The invention is embodied in a method of integrating kinematics equations for updating a set of vehicle attitude angles of a vehicle using 3-dimensional angular velocities of the vehicle, which includes computing an integrating factor matrix from quantities corresponding to the 3-dimensional angular velocities, computing a total integrated angular rate from the quantities corresponding to a 3-dimensional angular velocities, computing a state transition matrix as a sum of (a) a first complementary function of the total integrated angular rate and (b) the integrating factor matrix multiplied by a second complementary function of the total integrated angular rate, and updating the set of vehicle attitude angles using the state transition matrix. Preferably, the method further includes computing a quanternion vector from the quantities corresponding to the 3-dimensional angular velocities, in which case the updating of the set of vehicle attitude angles using the state transition matrix is carried out by (a) updating the quanternion vector by multiplying the quanternion vector by the state transition matrix to produce an updated quanternion vector and (b) computing an updated set of vehicle attitude angles from the updated quanternion vector. The first and second trigonometric functions are complementary, such as a sine and a cosine. The quantities corresponding to the 3-dimensional angular velocities include respective averages of the 3-dimensional angular velocities over plural time frames. The updating of the quanternion vector preserves the norm of the vector, whereby the updated set of vehicle attitude angles are virtually error-free.

21 Claims, 9 Drawing Sheets

Homogeneous transformations

$$\mathrm{Trans}_{x,a} = \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad Rot_{x,\alpha} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_\alpha & -s_\alpha & 0 \\ 0 & s_\alpha & c_\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathrm{Trans}_{y,b} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad Rot_{y,\beta} = \begin{bmatrix} c_\beta & 0 & s_\beta & 0 \\ 0 & 1 & 0 & 0 \\ -s_\beta & 0 & c_\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathrm{Trans}_{z,c} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad Rot_{x,\gamma} = \begin{bmatrix} c_\gamma & -s_\gamma & 0 & 0 \\ s_\gamma & c_\gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Homogeneous transformations

Composition Rule for Homogeneous Transformations

The same interpretation regarding composition and ordering of transformations holds for 4×4 homogeneous transformations as for 3×3 rotations. Given a homogeneous transformation H_1^0 relating two frames, if a second rigid motion, represented by $H \in SE(3)$ is performed relative to the current frame, then

$$H_2^0 = H_1^0 H$$

whereas if the second rigid motion is performed relative to the fixed frame, then

$$H_2^0 = H H_1^0$$

Comparison for different operations

Performance comparison of rotation chaining operations

Method	Storage	# multiplies	# add/subtracts	total operations
Rotation matrix	9	27	18	45
Quaternions	4	16	12	28

Performance comparison of various rotation operations

Method	Storage	# multiplies	# add/subtracts	# sin/cos	total operations
Rotation matrix	9	9	6	0	15
Quaternions	4	21	18	0	39
Angle/axis	4*	23	16	2	41

Further studies

- R.M. Murray, Z. Li, and S.S. Sastry: **A mathematical introduction to Robotic Manipulation** (Chapter 2)
- James Diebel: **Representing Attitude: Euler Angles, Unit Quaternions, and Rotation Vectors**
- Erik B. Dam, Martin Koch, and Martin Lillholm: **Quaternions, Interpolation and Animation**