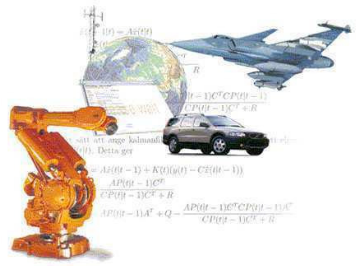
Nonlinear control

Lecture 1



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Overview of the course

- Make the control problem linear. 1 – 3
- Make the system stable: Lyapunov methods. 4 – 5
- Make the system optimal: 6
- Use physics: 8
- Use nonlinear feedforward

Examination

- Two homework assignments
  - Construct and simulate a nonlinear observer.
  - Construct and simulate a backstepping-based controller.
- A two-day take-home exam.

These tasks give 9 hp.

- A voluntary project gives a further 3 hp.

Nonlinear Control. Lecture 1

- Models of nonlinear systems
- Properties of differential equations
- Output control and exact output linearization

Models of nonlinear systems

Most common model:

$$\dot{x} = f(x, u), \quad y = h(x)$$

x state vector
u input vector
y output vector

Important special case:

$$\dot{x} = f(x) + g(x)u, \quad y = h(x)$$

Properties of differential equations

$$\dot{x} = f(t, x), \quad x(t_0) = x_0$$

1. f continuous $\Rightarrow$ solution exists locally
2. f Lipschitz-continuous $\Rightarrow$ solution
   a exists locally
   b is unique
   c depends Lipschitz-continuously on initial condition
3. f continuously differentiable $\Rightarrow$ (a), (b), (c) and solution depends cont. differentiably on initial condition

Properties of differential equations

To show global existence one has to show boundedness of the solution.
If f is infinitely differentiable, then the solution of

$$\dot{x} = f(x), \quad x(0) = x_0$$

can formally be written as

$$x(t) = x_0 + \sum_{k=1}^{\infty} \frac{t^k}{k!} f^{(k-1)}(x_0)$$

where $f^{(k)}$ is defined recursively:

$$f^{(k)}(x) = f_x^{(k-1)}(x) f(x), \quad f^{(0)}(x) = f(x)$$

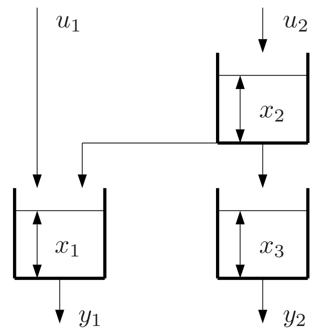
The Servo Problem

The basic servo problem:
for the system

$$\dot{x} = f(x) + g(x)u, \quad y = h(x)$$

choose u so that y equals ("as well as possible") a given reference signal r.

An Example: Coupled tanks



A model is

$$
\begin{aligned}
\dot{x}_1 &= -\sqrt{x_1} + \sqrt{x_2} + u_1 \\
\dot{x}_2 &= -2\sqrt{x_2} + u_2 \\
\dot{x}_3 &= -\sqrt{x_3} + \sqrt{x_2} \\
y_1 &= x_1 \\
y_2 &= x_3
\end{aligned}
$$

A controller for the tank system

The controller

$$
\begin{aligned}
u_1 &= \sqrt{x_1} - \sqrt{x_2} - ay_1 + ar_1 \\
u_2 &= \frac{\sqrt{x_2}}{\sqrt{x_3}}(\sqrt{x_2} - \sqrt{x_3}) + 2\sqrt{x_2} + 2\sqrt{x_2}(a_2 r_2 - a_1 \dot{y}_2 - a_2 y_2)
\end{aligned}
$$

gives the following decoupled dynamics

$$
\begin{aligned}
\dot{y}_1 + ay_1 &= ar_1 \\
\ddot{y}_2 + a_1 \dot{y}_2 + a_2 y_2 &= a_2 r_2
\end{aligned}
$$

Since a, a_1, a_2 are free to choose the response from r_i to y_i can be arbitrarily good.

Can this design technique be generalized?

Lie derivative

To handle multiple differentiations of the output for a system

$$\dot{x} = f(x), \quad y = h(x)$$

it is convenient to introduce the **Lie derivative** in direction f:

$$L_f = \sum_{i=1}^{n} f_i(x) \frac{\partial}{\partial x_i}$$

Then

$$\dot{y} = L_f h, \quad \ddot{y} = L_f(L_f h) = L_f^2 h, \quad \text{etc.}$$

Input-output linearization

$$\dot{x} = f(x) + g(x)u, \quad y = h(x)$$

dim y = dim u = m. Assume there exists smallest integer ν_i such that $y_i^{(\nu_i)}$ depends explicitly on u.
Decoupling matrix:

$$R(x) = \begin{bmatrix} L_{g_1} L_f^{\nu_1 - 1} h_1 & \dots & L_{g_m} L_f^{\nu_1 - 1} h_1 \\ \vdots & & \vdots \\ L_{g_1} L_f^{\nu_m - 1} h_m & \dots & L_{g_m} L_f^{\nu_m - 1} h_m \end{bmatrix}$$

If $R(x_0)$ nonsingular, the system is said to have **relative degree** $(\nu_1, \dots, \nu_m)$ at x_0.

Input-output linearization if R nonsingular

For the system

$$\begin{bmatrix} y_1^{(\nu_1)} \\ \vdots \\ y_m^{(\nu_m)} \end{bmatrix} = R(x) \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix} + \begin{bmatrix} L_f^{\nu_1} h_1 \\ \vdots \\ L_f^{\nu_m} h_m \end{bmatrix}$$

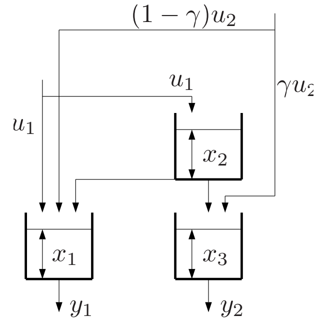
the state feedback

$$u = R(x)^{-1} \left(- \begin{bmatrix} L_f^{\nu_1} h_1 \\ \vdots \\ L_f^{\nu_m} h_m \end{bmatrix} + \begin{bmatrix} -a_{11} y_1^{(\nu_1 - 1)} - a_{1\nu_1} y_1 + a_{1\nu_1} r_1 \\ \vdots \\ -a_{m1} y_m^{(\nu_m - 1)} - a_{m\nu_m} y_m + a_{m\nu_m} r_m \end{bmatrix} \right)$$

gives linear decoupled systems

$$y_j^{(\nu_j)} + a_{j1} y_j^{(\nu_j - 1)} + \cdots + a_{j\nu_j} y_j = a_{j\nu_j} r_j, \quad j = 1, \ldots, m$$

Example: Coupled tanks, modified



A model is

$$\begin{aligned} \dot{x}_1 &= -\sqrt{x_1} + \sqrt{x_2} + u_1 + (1-\gamma)u_2 \\ \dot{x}_2 &= -2\sqrt{x_2} + u_1 \\ \dot{x}_3 &= -\sqrt{x_3} + \sqrt{x_2} + \gamma u_2 \\ y_1 &= x_1 \\ y_2 &= x_3 \end{aligned}$$

Controller for the modified tanks

For $\gamma = 0.1$ the controller

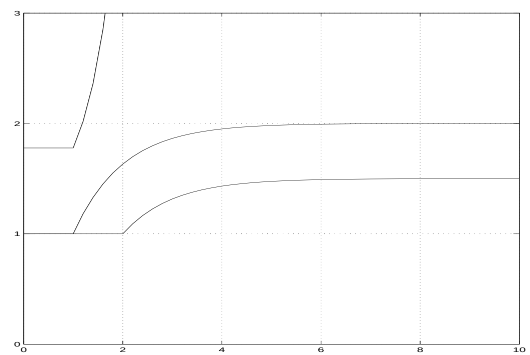
$$\begin{aligned} u_1 &= \sqrt{x_1} + 8\sqrt{x_2} - 9\sqrt{x_3} + 9a_2 x_3 - 9a_2 r_2 - a_1 x_1 + a_1 r_1 \\ u_2 &= 10(\sqrt{x_3} - \sqrt{x_2} - a_2 x_3 + a_2 r_2) \end{aligned}$$

gives the following closed loop dynamics

$$\begin{aligned} \dot{x}_1 &= -a_1 x_1 + a_1 r_1 \quad \Leftrightarrow \quad \dot{y}_1 = -a_1 y_1 + a_1 r_1 \\ \dot{x}_2 &= 6\sqrt{x_2} + f(x_1, x_3, r_1, r_2) \\ \dot{x}_3 &= -a_2 x_3 + a_2 r_2 \quad \Leftrightarrow \quad \dot{y}_2 = -a_2 y_2 + a_2 r_2 \end{aligned}$$

Step response of closed loop system. $\gamma = 0.1$

Steps in r_1 at $t = 1$ (ampl. 2) and in r_2 at $t = 2$ (ampl. 1.5)
y_1, y_2, x_2 are plotted.



Conclusion: Nice input-output behavior $\not\Rightarrow$ internal stability.

Controller for the modified tanks, cont'd

For $\gamma = 0.9$ the controller

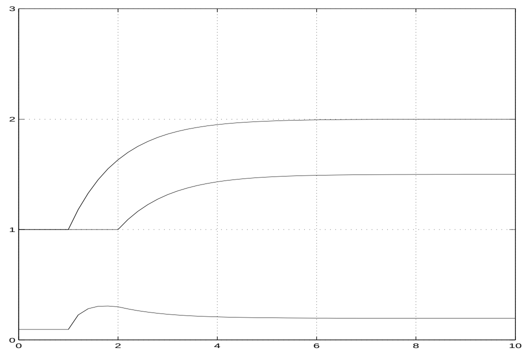
$$u_1 = \sqrt{x_1} + (\sqrt{x_2} - \sqrt{x_3} + a_2 x_3 - a_2 r_2)/9 - a_1 x_1 + a_1 r_1$$
$$u_2 = 10(\sqrt{x_3} - \sqrt{x_2} - a_2 x_3 + a_2 r_2)/9$$

gives the following closed loop dynamics

$$\dot{x}_1 = -a_1 x_1 + a_1 r_1 \quad \Leftrightarrow \quad \dot{y}_1 = -a_1 y_1 + a_1 r_1$$
$$\dot{x}_2 = -2.89\sqrt{x_2} + f(x_1, x_3, r_1, r_2)$$
$$\dot{x}_3 = -a_2 x_3 + a_2 r_2 \quad \Leftrightarrow \quad \dot{y}_2 = -a_2 y_2 + a_2 r_2$$

Step response of closed loop system. $\gamma = 0.9$

Steps in r_1 at $t = 1$ (ampl. 2) and in r_2 at $t = 2$ (ampl. 1.5)
y_1, y_2, x_2 are plotted.



Conclusion: The hidden dynamics might also be nice

Zero dynamics

The input-output dynamics has $\nu_1 + \cdots + \nu_m$ state variables.

There are n state variables in the system.

If $\nu_1 + \cdots + \nu_m < n$ the remaining state variables form the **zero dynamics**.

This is the dynamics that remains when all outputs are held constant.

Coordinate change: Introduce $y_1, \dot{y}_1, \ldots, y_1^{(\nu_1 - 1)}, \ldots, y_m, \dot{y}_m, \ldots, y_m^{(\nu_m - 1)}$ as new variables.

Nonlinear zero dynamics

The zero dynamics of a nonlinear system

$$\dot{x} = f(x, u), \quad y = h(x)$$

is given by

$$\dot{x} = f(x, u), \quad r = h(x), \quad r \text{ constant}$$

Interpretation: this is the dynamics that is left if perfect control of the output is achieved.

Linear zero dynamics

For a linear system the zero dynamics is given by

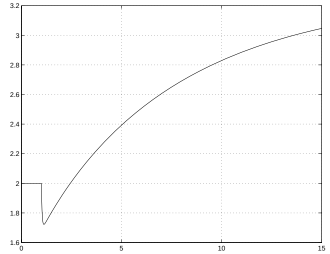
$$\dot{x} = Ax + Bu$$
$$0 = Cx$$

A solution $x = e^{\lambda t}x_o$, $u = e^{\lambda t}u_o$ belongs to the zero dynamics if

$$\begin{bmatrix} \lambda I - A & B \\ -C & 0 \end{bmatrix} \begin{bmatrix} -x_o \\ u_o \end{bmatrix} = 0$$

Thus λ is a **zero** of the linear system.
Zero in the right half plane $\Rightarrow$ unstable zero dynamics.

Time domain properties



For a nonlinear SISO system the following holds:
“If the linearized zero dynamics has an odd number of eigenvalues in the right half plane, then the step response has an undershoot”

(“Step responses of nonlinear non-minimum phase systems”, NOLCOS 2004, pp 1445-1449, LiTH-ISY-R-2586, Jan 2004)