Control of Systems Integrating Logic, Dynamics, and Constraints

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Goals:

Find an *optimal control* algorithm for a large class of hybrid (or almost hybrid) systems, including

- Piecewise linear systems
- Linear systems with bounded output
- Linear/Bilinear systems with discrete input
- Linear systems with quantized output
- Finite automata driven by linear systems
- Other (not mentioned) systems.

Use it for *soft constraints* and *predictive control* as well.

Idea:

Write the systems on a form suitable for an optimization algorithm, MILP/MIQP (*Mixed-Integer Linear/Quadratic Programming*).

The form used is called MLD (*Mixed Logical Dynamical*) form:

$$
\begin{aligned}
x(t+1) &= A_t x(t) + B_{1t} u(t) + B_{2t} \delta(t) + B_{3t} z(t) \\
y(t) &= C_t x(t) + D_{1t} u(t) + D_{2t} \delta(t) + D_{3t} z(t) \\
-E_{5t} &\preccurlyeq E_{4t} x(t) + E_{1t} u(t) - E_{2t} \delta(t) - E_{3t} z(t)
\end{aligned}
$$

Properties:

- Linear equations containing *both continuous and discrete* variables.
- Variables constrained by linear inequalities.
- Time discrete system.

Technicalities about the MLD form:

- An MLD system is *well posed* if, for all t, $x(t+1)$ and the $\delta(t)$, $z(t)$ occuring in the two equations are uniquely defined by $x(t)$, $u(t)$.
- It is *completely well posed* if it is well posed and all $\delta(t)$, $z(t)$ occur in (at least one of) the two equations.

If the MLD system is well posed, it is just a complicated way of writing a system

$$\begin{aligned} x(t+1) &= f(x(t), u(t), t) \\ y(t) &= g(x(t), u(t), t) \end{aligned}$$

for a certain class of functions f, g.

Outline for the rest of the talk:

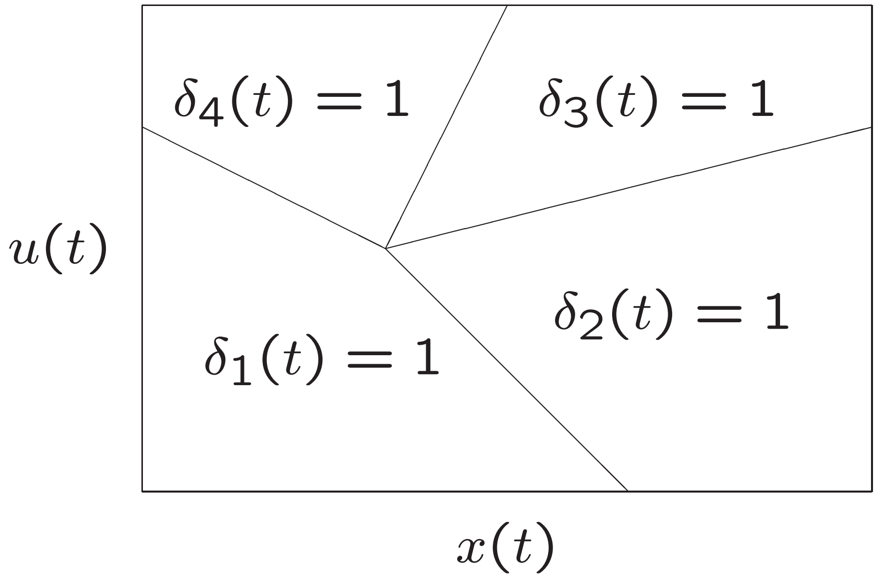
- Translate piecewise linear system into MLD system.
- Optimal control problem.
- Solving optimal control problem using MIQP.

Piecewise linear system

$$x(t+1) = \begin{cases} A_1 x(t) + B_1 u(t) & \text{if } \delta_1(t) = 1 \\ \vdots & \\ A_s x(t) + B_s u(t) & \text{if } \delta_s(t) = 1 \end{cases}$$

(we can also have, e.g., $y(t) = x(t)$).

$\delta_i(t) = 1$ iff $\begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \in \mathcal{C}_i$, a region defined by $S_i x + R_i u \preccurlyeq T_i$.



Rewrite in MLD form:

- Equation for $x(t+1)$:

$$x(t+1) = \sum_{i=1}^{s} (A_i x(t) + B_i u(t))\delta_i(t)$$

- Only one $\delta_i = 1$:

$$\sum_{i=1}^{s} \delta_i(t) = 1$$

- $\delta_i(t) = 1$ when $\begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \in \mathcal{C}_i$:

$$S_i x(t) + R_i u(t) - T_i \preccurlyeq M_i^*(1 - \delta_i(t))$$

We're not ready yet!

- Equation for $x(t+1)$ is nonlinear, so let

$$z_i(t) = (A_i x(t) + B_i u(t))\delta_i(t)$$

which gives $x(t+1) = \sum z_i(t)$.

- Now the definition of $z_i(t)$ is nonlinear, but we can rewrite it as

$$\begin{aligned}
z_i(t) &\preccurlyeq M\delta_i(t) \\
z_i(t) &\succcurlyeq m\delta_i(t) \\
z_i(t) &\preccurlyeq A_i x(t) + B_i u(t) - m(1 - \delta_i(t)) \\
z_i(t) &\succcurlyeq A_i x(t) + B_i u(t) - M(1 - \delta_i(t))
\end{aligned}$$

Now we are ready!

Optimal control problem

Control problem: Try to drive x to x_f, i.e., find $u_0^{T-1} = u(0), \ldots, u(T-1)$ so that $x(T) = x_f$ for some T.

Doing it optimally: Minimise

$$\begin{aligned} J(u_0^{T-1}, x_0) = \sum_{t=0}^{T-1} & \|u(t) - u_f\|_{Q_1}^2 + \|\delta(t) - \delta_f\|_{Q_2}^2 \\ & + \|z(t) - z_f\|_{Q_3}^2 + \|x(t) - x_f\|_{Q_4}^2 \\ & + \|y(t) - y_f\|_{Q_5}^2 \end{aligned}$$

subject to $x(T) = x_f$ and the MLD dynamics.

Interpretation?

Optimal control problem is a MIQP problem

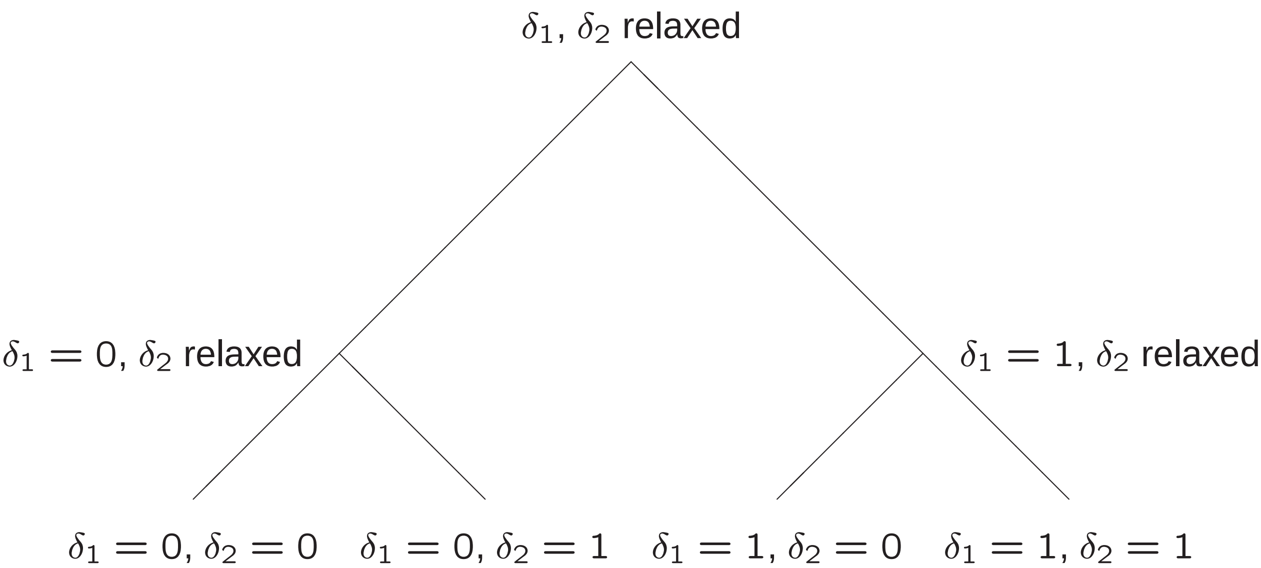
The optimal control problem is a problem of the following type:

$$\begin{aligned} \min_{x,\delta} \quad & \begin{pmatrix} x^T & \delta^T \end{pmatrix} Q \begin{pmatrix} x \\ \delta \end{pmatrix} + p^T \begin{pmatrix} x \\ \delta \end{pmatrix} \\ \text{subj. to} \quad & C \begin{pmatrix} x \\ \delta \end{pmatrix} \leq d \\ & \delta \in \{0, 1\}^m \end{aligned}$$

(Here, x are all the continuous variables and δ all discrete variables!)

This is a *Mixed-Integer Quadratic Program* (MIQP).

Solving MIQP via Branch-and-Bound



Extensions of optimal control

- *Soft constraints:* Replace constraint $Ax \preccurlyeq B$ with

$$Ax - C\varepsilon \preccurlyeq B$$

  and add a penalty term $\varepsilon^T M\varepsilon$ to the objective function.

- *Predictive control:* Solve an optimal control problem in each time step, and use the first control input obtained. (Can be relaxed: we do not need to solve the optimal control problem completely – it is sufficient that the objective function decreases in each time step.)