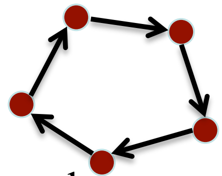
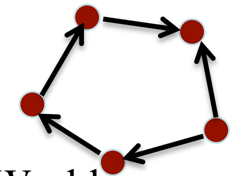
SESSION 2
MULTI-AGENT NETWORKS

Variations on the Theme: Directed Graphs

- Instead of connectivity, we need directed notions:
  - ***Strong connectivity*** = there exists a directed path between any two nodes
  - ***Weak connectivity*** = the disoriented graph is connected



Strongly connected



Weakly connected

- Directed consensus:

$$\dot{x}_i = -\sum_{j \in N_i^{in}} (x_i - x_j)$$

Directed Consensus

- Undirected case: Graph is connected = sufficient information is flowing through the network
- Clearly, in the directed case, if the graph is strongly connected, we have the same result

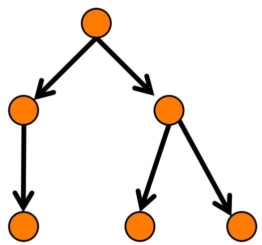
- **Theorem**: If G is strongly connected, the consensus equation achieves

$$\lim_{t \to \infty} (x_i - x_j) = 0, \ \forall i, j$$

- This is an unnecessarily strong condition! Unfortunately, weak connectivity is too weak.

Rooted Outbranching Trees

- Consider the following structure



- Seems like all agents should end up at the root node

- **Theorem [2]**: Consensus in a directed network is achieved if and only if G contains a spanning rooted outbranching tree (ROT).

Where Do the Agents End Up?

- Recall: Undirected case

$$\lim_{t\to\infty} x_i(t) = \bar{x}(0) = \frac{1}{N}\sum_{j=1}^{N} x_j(0),\ \forall i$$

- How show that?
- The centroid is invariant under the consensus equation

$$\dot{\bar{x}} = \frac{1}{N}\sum_{i=1}^{N}\sum_{j\in N_i}(x_j - x_i) = 0$$

- And since the agents end up at the same location, they must end up at the static centroid (average consensus).

Where Do the Agents End Up?

- When is the centroid invariant in the directed case?

$$q^T L = 0,\ w = q^T x \ \Rightarrow\ \dot{w} = q^T \dot{x} = -q^T L x = 0$$

- w is invariant under the consensus equation
- The centroid is given by

$$\bar{x} = \frac{1}{N}\mathbf{1}^T x$$

which thus is invariant if

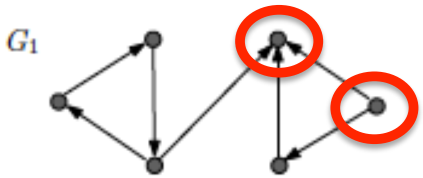
$$\mathbf{1}^T L = 0$$

- **Def**: G is balanced if

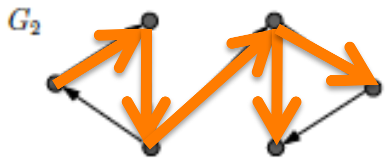
$$deg^{in}(i) = deg^{out}(i),\ \forall i \in V \ \Leftrightarrow\ \mathbf{1}^T L = 0$$

- **Theorem [2]:** If G is balanced and consensus is achieved then average consensus is achieved!

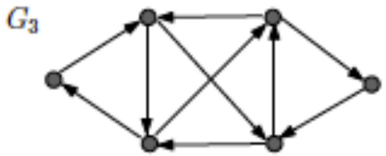
Example



No ROT – Consensus is not achieved



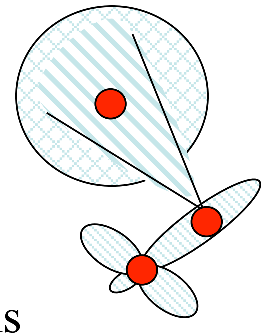
ROT but not balanced – Consensus but not average consensus is achieved

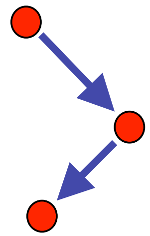


ROT and balanced – Average consensus is achieved

Beyond Static Consensus

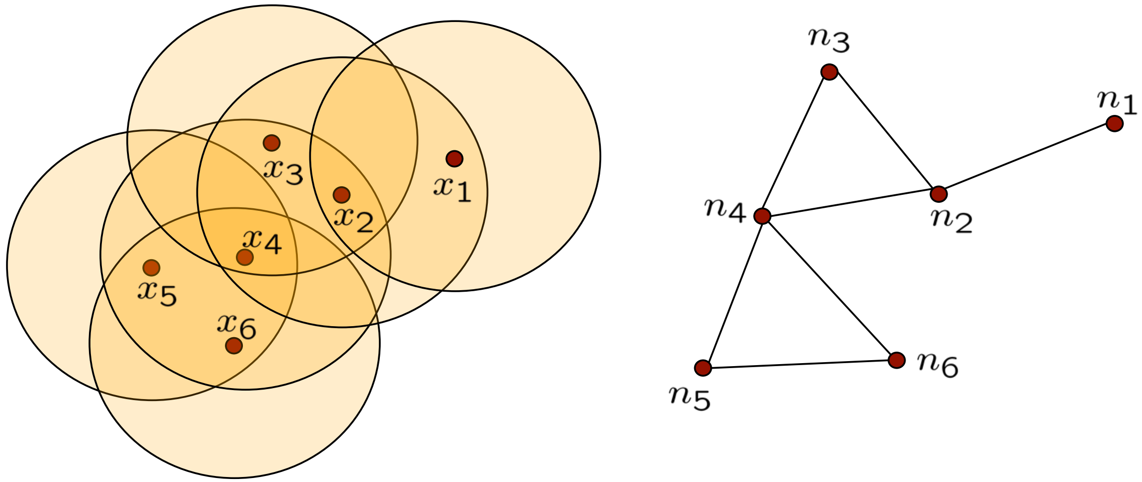
- So far, the consensus equation will drive the node states to the same value if the graph is static and connected.
- But, this is clearly not the case in a number of situations:
  - **Edges = communication links**
    - Random failures
    - Dependence on the position (shadowing,…)
    - Interference
    - Bandwidth issues
  - **Edges = sensing**
    - Range-limited sensors
    - Occlusions
    - Weirdly shaped sensing regions





Dynamic Graphs

- In most cases, edges correspond to available sensor or communication data, i.e., the edge set is time varying



- We now have a switched system and spectral properties do not help for establishing stability
- Need to use Lyapunov functions

Lyapunov Functions

- Given a nonlinear system

$$\dot{x} = f(x)$$

- V is a (weak) Lyapunov function if

$$(i) \quad V(x) > 0, \ \forall x \neq 0$$
$$(ii) \quad \dot{V}(x) = \frac{\partial V}{\partial x} f(x) < 0, \ \forall x \neq 0 \quad (\leq 0)$$

- The system is asymptotically stable if and only if there exists a Lyapunov function
- [LaSalle's Invariance Principle] If it has a weak Lyapunov function the system converges asymptotically to the largest invariant set (f=0) s.t. the derivative is 0

Switched Systems

- Similarly, consider a switched system

$$\dot{x} = f_\sigma(x), \quad \sigma(t) \in \{1, \ldots, q\}$$

- The system is ***universally asymptotically stable*** if it is asymptotically stable for all switch sequences

- A function V is a common Lyapunov function if it is a Lyapunov function to all subsystems

$$V > 0, \ \frac{\partial V}{\partial x} f_i < 0, \ i = 1, \ldots, q$$

- **Theorem [9]**: Universal stability if and only if there exists a common Lyapunov function. (Similar for LaSalle.)

Switched Networked Systems

- Switched consensus equation

$$\dot{x} = -L_\sigma x$$

- Consider the following candidate Lyapunov function

$$V(x) = \frac{1}{2}x^T x, \quad \dot{V}(x) = x^T \dot{x} = -x^T L_\sigma x$$

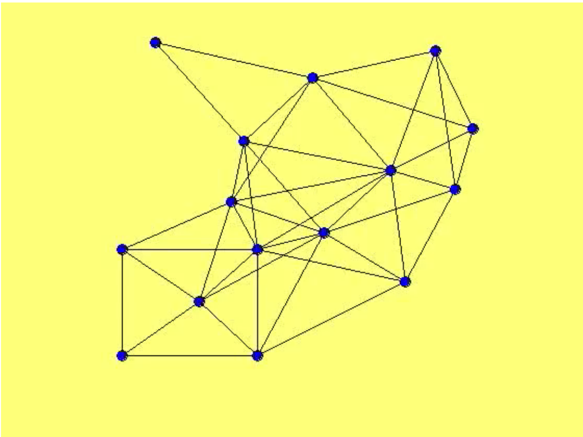
- This is a common (weak) Lyapunov function as long as *G* is connected for all times

- Using LaSalle's theorem, we know that in this case, it ends up in the null-space of the Laplacians

Switched Consensus

Theorem [2-4]: As long as the graph stays connected, the ***consensus equation*** drives all agents to the same state value

$$\lim_{t\to\infty} x_i(t) = \bar{x} = \frac{1}{N}\sum_{j=1}^{N} x_j(0)$$

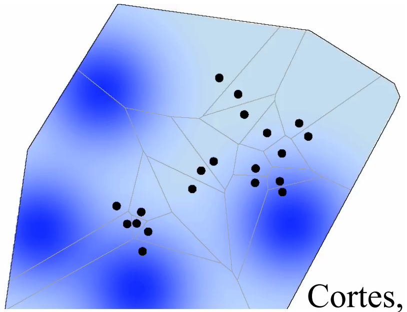


Adding Weights

- Sometimes it makes sense to add weights

$$\dot{x}_i = -\sum_{j \in N_i} w(\|x_i - x_j\|)(x_i - x_j)$$

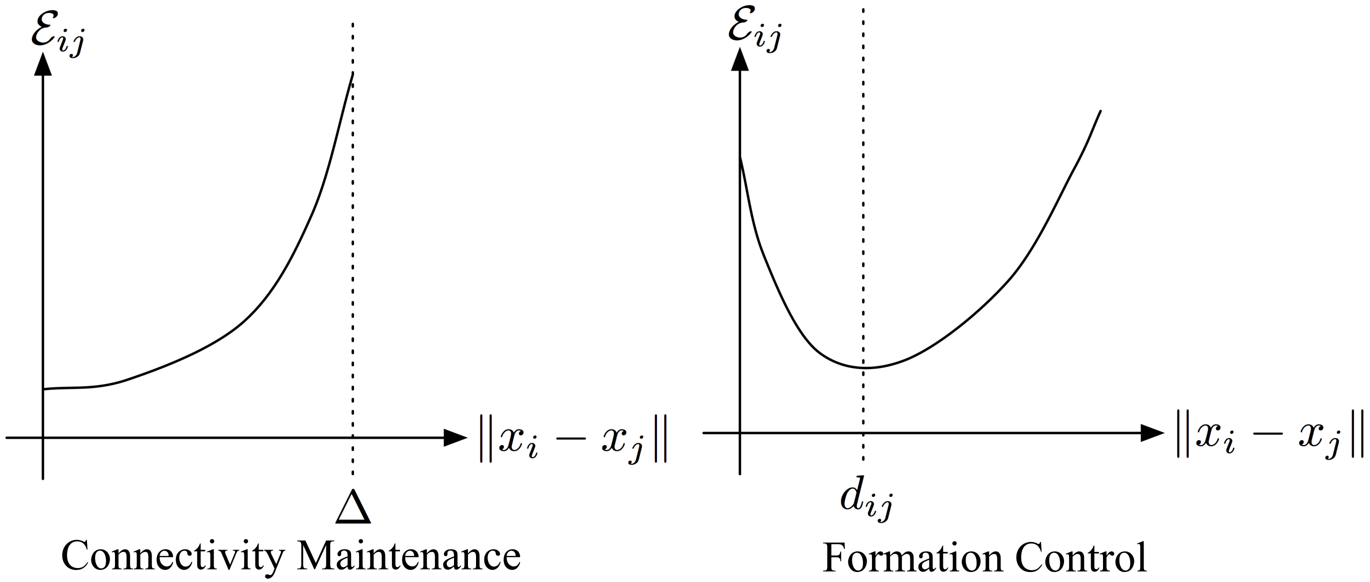
- Collision avoidance
- Coverage
- Connectivity maintenance



Cortes, Martinez, Bullo

Weights Through Edge Tensions

- How select appropriate weights?
- Let an edge tension be given by $\mathcal{E} = \sum_{i=1}^{N}\sum_{j=1}^{N} a_{i,j}\mathcal{E}_{i,j}(\|x_i - x_j\|)$



Weights Through Edge Tensions

- How select appropriate weights?
- Let an edge tension be given by $\mathcal{E} = \sum_{i=1}^{N} \sum_{j=1}^{N} a_{i,j} \mathcal{E}_{i,j}(\|x_i - x_j\|)$
- We get

$$\frac{\partial \mathcal{E}_{i,j}}{\partial x_i} = w_{i,j}(\|x_i - x_j\|)(x_i - x_j)$$

- Gradient descent

$$\dot{x}_i = -\frac{\partial \mathcal{E}}{\partial x_i} = -\sum_{j \in N_i} w_{i,j}(\|x_i - x_j\|)(x_i - x_j)$$

$$\frac{d\mathcal{E}}{dt} = \frac{\partial \mathcal{E}}{\partial x}\dot{x} = -\left\| \frac{\partial \mathcal{E}}{\partial x} \right\|^2$$

Energy is non-increasing!
(weak Lyapunov function)

Examples

$$\mathcal{E}_{ij} = \frac{1}{2}\|x_i - x_j\|^2 \;\Rightarrow\; w_{ij} = 1$$

$$\dot{x}_i = -\sum_{j \in N_i} (x_i - x_j)$$

Standard, linear consensus!

$$\mathcal{E}_{ij} = \|x_i - x_j\| \;\Rightarrow\; w_{ij} = \frac{1}{\|x_i - x_j\|}$$

$$\dot{x}_i = -\sum_{j \in N_i} \frac{x_i - x_j}{\|x_i - x_j\|}$$

Unit vector (biology)

Examples

$$\mathcal{E}_{ij} = \frac{1}{2}(\|x_i - x_j\| - d_{ij})^2 \;\Rightarrow\; w_{ij} = \frac{\|x_i - x_j\| - d_{ij}}{\|x_i - x_j\|}$$

$$\dot{x}_i = -\sum_{j \in N_i} \frac{(\|x_i - x_j\| - d_{ij})(x_i - x_j)}{\|x_i - x_j\|}$$

Formation control

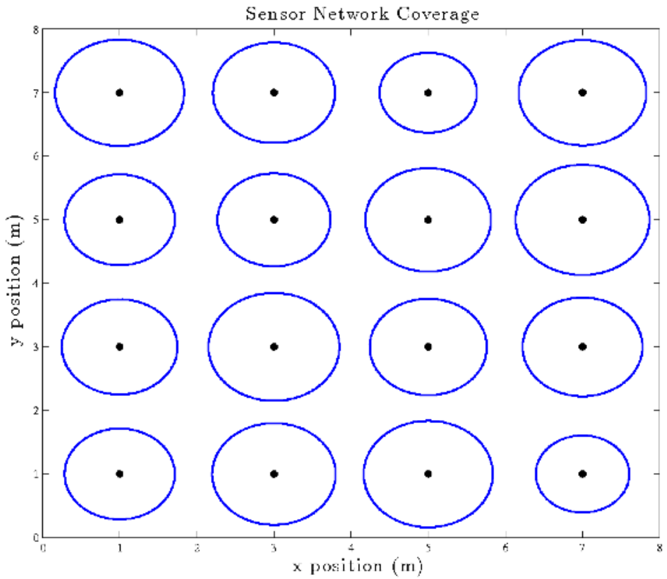
$$\mathcal{E}_{ij} = \frac{\|x_i - x_j\|^2}{\Delta - \|x_i - x_j\|} \;\Rightarrow\; w_{ij} = \frac{2\Delta - \|x_i - x_j\|}{(\Delta - \|x_i - x_j\|)^2}$$

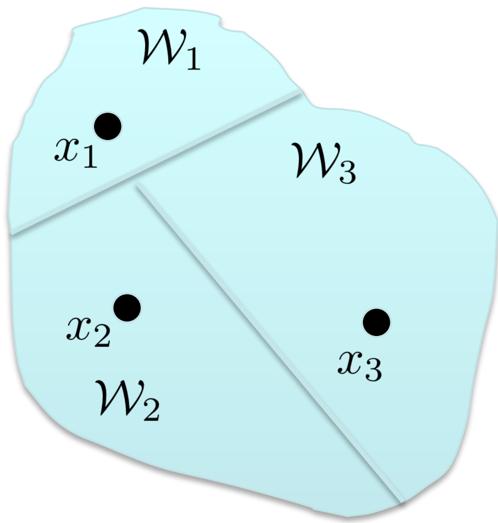
$$\dot{x}_i = -\sum_{j \in N_i} \frac{(2\Delta - \|x_i - x_j\|)(x_i - x_j)}{(\Delta - \|x_i - x_j\|)^2}$$

Connectivity maintenance

Coverage Control

- Objective: Deploy sensor nodes in a distributed manner such that an area of interest is covered





- Idea: Divide the responsibility between nodes into regions

Coverage Control

- The coverage cost:

$$J(x,\mathcal{W}) = \frac{1}{2}\sum_{i=1}^{N}\int_{\mathcal{W}_i} \|x_i - q\|^2 dq$$

- Simplify (not optimal):

$$\hat{J}(x) = \frac{1}{2}\sum_{i=1}^{N}\int_{\mathcal{V}_i(x)} \|x_i - q\|^2 dq$$

where the Voronoi regions are given by

$$\mathcal{V}_i(x) = \{q \in \mathcal{D} \mid \|x_i - q\| \leq \|x_j - q\|\}$$

Deployment

- Using a gradient descent (cost = weak Lyapunov function)

$$\dot{x}_i = -\frac{\partial \hat{J}}{\partial x_i} \;\Rightarrow\; \frac{d}{dt}\hat{J} - \left\| \frac{\partial \hat{J}}{\partial x} \right\|^2$$

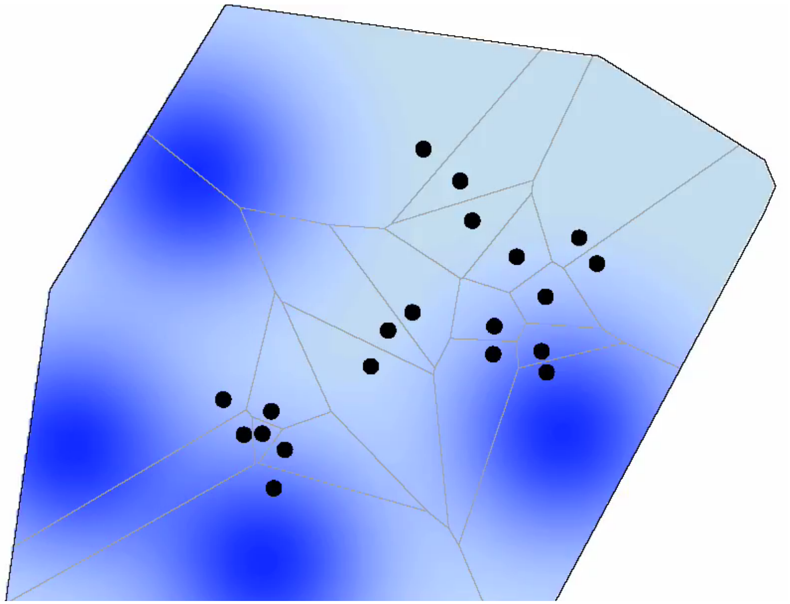
$$\dot{x}_i = -\int_{\mathcal{V}_i(x)} (x_i - q)dq$$

- We only care about directions so this can be re-written as Lloyd's Algorithm [1]

$$\dot{x}_i = \rho_i(x) - x_i$$

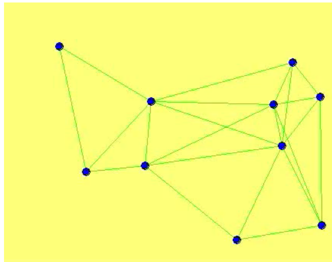
Deployment

- Lloyd's Algorithm:
  - Converges to a local minimum to the simplified cost
  - Converges to a Central Voronoi Tessellation
  - It is decentralized

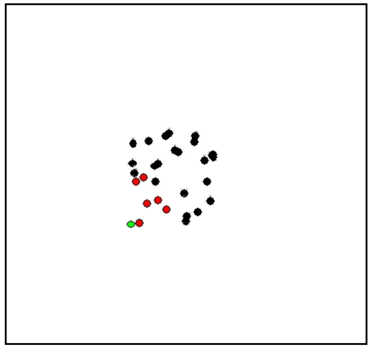


Graph-Based Control

- In fact, based on variations of the consensus equation, a number of different multi-agent problems have been “solved”, e.g.
  - **Formation control** (How drive the collection to a predetermined configuration? [2,5])
  - **Coverage control** (How produce triangulations or other regular structures? [1,6])



- ***OK – fine. Now what?***
- Need to be able to **reprogram and redeploy** multi-agent systems (**HSI = Human-Swarm Interactions**)
- This can be achieved through active control of so-called leader-nodes



Summary II

- Static Graphs:
    - Undirected: Average consensus iff G is connected
    - Directed: Consensus iff G contains a spanning, outbranching tree
    - Directed: Average consensus if consensus and G is balanced
- Switching Graphs:
    - Undirected: Average consensus if G is connected for all times
    - Directed: Consensus if G contains a spanning, outbranching tree for all times
    - Directed: Average consensus if consensus and G is balanced for all times
- Additional objectives is achieved by adding weights (edge-tension energies as weak Lyapunov functions)