Homework Assignment 1

The homework assignment should be solved *individually*. However, it is allowed to ask questions to fellow students. All solutions should be clearly written and well motivated. Regarding the language, you may use Swedish or English.

The assignment is due on Wednesday, December 10, 2008. Please email your solutions to schon@isy.liu.se.

Rigid Body Motion

1. *Axis/angle representation*
   The rotation matrix contains 9 parameters and is thus a highly redundant parameterization of $SO(3)$. According to Euler's theorem, any orientation $R \in SO(3)$ is equivalent to a rotation about a fixed axis $\mathbf{u} \in \mathbb{R}^3$ through an angle $\varphi \in [0, 2\pi)$. This suggests the axis/angle parameterization, which can be parameterized using four parameters.

   (a) In Figure 1 we provide the geometry of a rotation of a point N

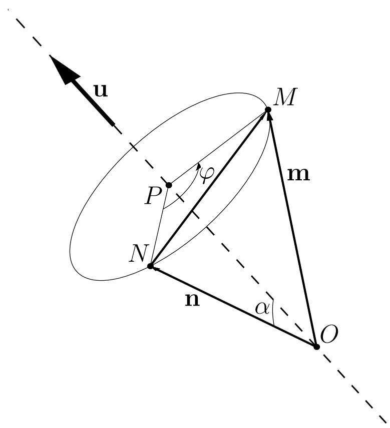

Figure 1: *Illustration of the axis/angle representation. The point N is rotated about the* $\mathbf{u}$*-axis through an angle* φ*.*

(indicated by the vector $\mathbf{n}$) about $\mathbf{u}$ through an angle φ. Using the geometry introduced in Figure 1, derive an expression for the rotation matrix

$$R = f(\mathbf{u}, \varphi) \tag{1}$$

corresponding to the rotation around the axis $\mathbf{u}$ through an angle φ. *Hint: The resulting equation is called Rodrigues' formula.*

   (b) Derive equations showing how to obtain the axis of rotation $\mathbf{u}$ and angle of rotation φ from the rotation matrix R.

   (c) A representation which only makes use of three parameters can be obtained by recalling that there is only two parameters used for representing a direction, the last one is given by the unit constraint.

This leads to the so called rotation vector

$$\mathbf{r} = \begin{pmatrix} \varphi u_x \\ \varphi u_y \\ \varphi u_z \end{pmatrix} = \varphi \mathbf{u}, \tag{2}$$

where the length of the vector $\mathbf{r}$ provides the angle. Why is this representation used to parameterize the rotation in the camera calibration problem? (see (10) in Zhang (2000)).

2. *Properties of the rotation matrix*

   (a) (Exercise 2.11 from Ma et al. (2006), repeated here for convenience) Let $R \in SO(3)$ be a rotation matrix generated by rotating about a unit vector ω by θ radians that satisfies $R = \exp(\hat{\omega}\theta)$. Suppose R is given as

   $$R = \begin{pmatrix} 0.1729 & -0.1468 & 0.9739 \\ 0.9739 & 0.1729 & -0.1468 \\ -0.1468 & 0.9739 & 0.1729 \end{pmatrix} \tag{3}$$

   Compute the rotation axis and the associated angle using the equation derived in 1b.

   Use MATLAB's function `eig` to compute the eigenvalues and eigenvectors of the above rotation matrix R. What is the eigenvector associated with the unit eigenvalue? Give its form and explain its meaning.

   (b) Let $R = \begin{pmatrix} r_1 & r_2 & r_3 \end{pmatrix}$ be a rotation matrix. Show that

   $$\det R = r_1^T (r_2 \times r_3). \tag{4}$$

   (c) (Exercise 2.10 from Ma et al. (2006), repeated here for convenience) What is the matrix that represents a rotation about the X- or the Y-axis by an angle θ? In addition to that

   i. Compute the matrix R_1 that is the combination of a rotation about the X-axis by $\pi/3$ followed by a rotation about the Z-axis by $\pi/6$. Verify that the resulting matrix is also a rotation matrix.

   ii. Compute the matrix R_2 that is the combination of a rotation about the Z-axis by $\pi/6$ followed by a rotation about the X-axis by $\pi/3$. Are R_1 and R_2 the same? Explain why/why not?

3. *Unit quaternions*
   Another commonly used representation when it comes to rotations is the unit quaternion,

   $$q = \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} q_0 \\ \mathbf{q} \end{pmatrix}, \qquad \|q\| = 1. \tag{5}$$

The unit quaternion is closely related to the axis/angle representation according to

$$q = \begin{pmatrix} \cos\frac{\varphi}{2} \\ u_x \sin\frac{\varphi}{2} \\ u_y \sin\frac{\varphi}{2} \\ u_z \sin\frac{\varphi}{2} \end{pmatrix} \tag{6}$$

which motivates the partitioning of the quaternion into a scalar part q_0 and a vector part $\mathbf{q}$ as we did in (5).

The dynamics describing the evolution of the unit quaternion over time is given by

$$\dot{q} = \frac{1}{2} S(\omega) q, \tag{7}$$

where ω denotes the angular velocity and $S(\omega)$ is a skew-symmetric matrix according to

$$S(\omega) = \begin{pmatrix} 0 & -\omega_x & -\omega_y & -\omega_z \\ \omega_x & 0 & \omega_z & -\omega_y \\ \omega_y & -\omega_z & 0 & \omega_x \\ \omega_z & \omega_y & -\omega_x & 0 \end{pmatrix} \tag{8}$$

(a) In order to implement an estimator we typically need an expression in discrete-time. Derive an expression for the dynamics in discrete-time using the Euler forward approximation of the derivative

$$\dot{q}(t) = \frac{q_{t+T} - q_t}{T}, \tag{9}$$

where T denotes the sampling time.

(b) Let us now investigate another alternative to arrive at an expression for the quaternion dynamics in discrete-time. The angular velocities are obtained at discrete instances. Hence, the inter-sample behaviour has to be postulated. Here, it will be assumed that ω is piece-wise constant between sampling instants, i.e.,

$$\omega(t) = \omega_{kT}, \qquad kT \le t < (k+1)T. \tag{10}$$

Derive an expression for the discrete-time dynamics based on this assumption.

(c) How are the two expressions for the discrete-time dynamics derived above related? More specifically, how can the expression derived in (a) be obtained from the expression derived in (b)?

(d) Before we give the assignment let us define a few quaternion properties. Quaternion multiplication,

$$p \odot q = \begin{pmatrix} p_0 \\ \mathbf{p} \end{pmatrix} \odot \begin{pmatrix} q_0 \\ \mathbf{q} \end{pmatrix} = \begin{pmatrix} p_0 q_0 - \mathbf{p}^T \mathbf{q} \\ p_0 \mathbf{q} + q_0 \mathbf{p} + \mathbf{p} \times \mathbf{q} \end{pmatrix}, \tag{11}$$

quaternion norm

$$\text{norm}(q) = \|q\| = \sqrt{\sum_{i=0}^{3} q_i^2} \tag{12}$$

and quaternion inverse

$$q^{-1} = \begin{pmatrix} q_0 \\ \mathbf{q} \end{pmatrix}^{-1} = \frac{1}{\|q\|^2} \begin{pmatrix} q_0 \\ -\mathbf{q} \end{pmatrix} \tag{13}$$

A rotation is, using quaternions, defined according to

$$\tilde{a}_\text{b} = q_\text{ab}^{-1} \odot \tilde{a}_\text{a} \odot q_\text{ab} \tag{14}$$

where $\tilde{a}$ is a quaternion with $q_0 = 0$, i.e.,

$$\tilde{a} = \begin{pmatrix} 0 & a_x & a_y & a_z \end{pmatrix}^T = \begin{pmatrix} 0 & \mathbf{a} \end{pmatrix}^T \tag{15}$$

Derive an expression for the rotation matrix R_ab, describing the rotation from frame b to frame a, in terms of the quaternion

$$q_{ab} = \begin{pmatrix} q_0 & q_1 & q_2 & q_3 \end{pmatrix}^T \tag{16}$$

Camera Models and Calibration

1. *Camera calibration*
   The purpose of this task is to get some hands on experience on using camera calibration software. Start by downloading and installing the Caltech camera calibration toolbox, see Bouguet (2008) for download and instructions. Background information: The camera used to acquire the images used in this exercise is a Point Grey Firefly MV. If you are interested, more information about this camera is available from this web site,

   `www.ptgrey.com/products/fireflymv/ads/may_2007/index.html`

   This is the camera used in the combined camera-IMU sensor unit showed during the first lecture, see Figure 2.

   (a) Download the data `camIMUdata.zip` from the course web site (both the images and the IMU data are available). Select 3 images to be used for camera calibration. What should you think about when you choose which images to use? Provide the estimated intrinsic and lens distortion parameters and their standard deviation. *Hint: Rename the images according to the convention used in the toolbox, e.g. image1.jpg, image2.jpg, etc.*

   (b) Extract 17 more images and redo the calibration procedure using all 20 images. Compare your result to the result obtained in 1a. Explain the differences/similarities.

Figure 2: *The combined camera and IMU sensor used to obtain the data.*

2. *Rectangle projection*
   The calibrated camera model is now to be used in order to project a rectangle into the image. Choose one of the following tasks (you can of course solve both)

   (a) Draw a rectangle around the phone on the desk, see Figure 3 for the position of the phone. In order to find the positions of the other corners you need to know that the phone is an iPhone. You task is now to draw a rectangle around the phone.

Figure 3: *Position of the iPhone. The origin is indicated using a circle and the two vectors given in the figure are* $\mathbf{u} = (220 \quad -118 \quad 0)^T$ *and* $\mathbf{v} = (151 \quad -90 \quad 0)^T$ *(in world coordinates, unit:* mm*).*

   (b) There is definitely a need for painting on the walls in the office where the images were acquired. Your task is to project a rectangle on the wall in front of the calibration pattern. The rectangle should be positioned 100 mm above the desk. The rectangle should be 150 mm wide and 100 mm high. The distance from the origin (as defined in

Figure 3) to the wall is 527 mm.

The squares are 30 mm by 30 mm. *Hint: The camera calibration software can be used to find the camera pose. Start by making sure that you can mark the corners of the checkerboard pattern.*

3. *Straight lines are straight*
Assume that we have a camera with negligible lens distortion (or that the lens distortion has been compensated for). This implies that the geometric model of the camera is given by (see e.g., (3.19) in Ma et al. (2006))

$$\lambda \begin{pmatrix} x_p \\ y_p \\ 1 \end{pmatrix} = \Pi \begin{pmatrix} x_w \\ y_w \\ z_w \\ 1 \end{pmatrix} \tag{17}$$

Now, show that the projection of a 3D line in the world coordinate frame is a line also in the pixel coordinate frame. All points on a given line in the world coordinate system can be parameterized according to

$$p_w = \bar{p}_w + \gamma u, \tag{18}$$

where $\bar{p}_w$ is a point on the line, u is a unit vector and $\gamma \in \mathbb{R}$ provides the distance along the line.

4. *Vanishing points*
In Exercise 3 you proved that a straight line in the 3D world is projected into a straight line in the image plane. Now, consider two parallel ($u^1 = u^2$) lines,

$$p_w = \bar{p}_w^1 + \gamma u^1, \tag{19a}$$
$$p_w = \bar{p}_w^2 + \gamma u^2. \tag{19b}$$

Show that the projection of two parallel lines intersect at a point in the image. This point is called the *vanishing point* and you have probably seen this in photographs like the one in Figure 4.

References

Bouguet, J.-Y. (2008). Camera calibration toolbox for Matlab. www.vision.caltech.edu/bouguetj/calib_doc.

Ma, Y., Soatto, S., Kosecka, J., and Sastry, S. S. (2006). *An invitation to 3-D vision – from images to geometric models.* Interdisciplinary Applied Mathematics. Springer.

Zhang, Z. (2000). A flexible new technique for camera calibration. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 22(11):1330–1334.

Figure 4: *Illustration of the concept of a vanishing point. This is the point where parallel lines intersect in the image. (This photo was taken during the CDC conference in New Orleans, USA in December* 2007.*)*